LES MODELING OF ANISOTROPIC MHD TURBULENCE

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ABSTRACT

Turbulent fluctuations in MHD flows can become strongly anisotropic or even quasi-two-dimensional under the action of an applied magnetic field. We investigate this phenomenon in the case of low magnetic Reynolds numbers. It has been found in earlier DNS and LES of homogeneous turbulence that the degree of anisotropy is predominantly determined by the value of the magnetic interaction parameter and only slightly depends on the Reynolds number, type of large-scale dynamics, and the length scale. Furthermore, it has been demonstrated that the dynamic Smagorinsky model is capable of self-adjustment to the effects of anisotropy. In this paper, we summarize the results and propose a simple and effective generalization of the traditional non-dynamic Smagorinsky model to the case of anisotropic MHD turbulence.

MHD TURBULENCE AT LOW MAGNETIC REYNOLDS NUMBER

Magnetohydrodynamic (MHD) turbulent flows occur in numerous astrophysical, geophysical, and technological applications. We consider the case of low magnetic Reynolds number

$$R_m \equiv uL/\eta \ll 1, \tag{1}$$

typical for the industrial and laboratory flows of liquid metals, oxide melts, and other electrically conducting fluids (see, e.g. Davidson (2001)). In (1), u and L are the typical velocity and length scales, and $\eta = (\sigma\mu_0)^{-1}$ is the magnetic diffusivity, σ and μ_0 being the electric conductivity of the liquid and the magnetic permeability of vacuum.

Low-R_m interaction between a static magnetic field and a turbulent flow is an important factor of some metallurgical operations, such as continuous steel casting or growth of large semiconductor crystals, where magnetic fields are intentionally used to non-intrusively suppress the unwanted development of the flow. In other cases, such as the primary aluminum production in Hall-Héroult or inert anode processes, or in the lithium cooling blankets for magnetic confinement fusion, the inevitably present static magnetic field has an adverse effect on performance, which has to be minimized through optimization of the process.

The results of the present work, albeit rigorously valid only in the case of low R_m, can be extended to the situations with moderate R_m and high hydrodynamic Reynolds number, most notably to the earth dynamo problem. The diffusive cut-off scale of the magnetic field is sufficiently large in such cases so that the magnetic field can be fully resolved in the DNS-like manner. The task of modeling the turbulent velocity fluctuations at smaller scales reduces to the problem addressed in this paper.

In the low-R_m case, the MHD equations can be significantly simplified by applying the quasi-static approximation. The perturbations of the magnetic field induced by fluid motions are small in comparison with the imposed magnetic field and can be neglected. They can also be approximately assumed to adjust instantaneously to the velocity perturbations. The Lorentz force is expressed as a linear functional of velocity and the governing equations can be represented in a closed form as

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{u} - \sigma B^2\rho^{-1}\Delta^{-1}\partial_{zz}\mathbf{u}, \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3}$$

where we assume that the magnetic field is $\mathbf{B} = B_0\mathbf{e}_z$, and Δ^{-1} is the reciprocal Laplace operator that stands for a solution of the Poisson equation for the electric potential with proper boundary conditions. The non-dimensional form of (2) contains two dimensionless parameters, one of which is the Reynolds number $Re \equiv uL/\nu$ and another is the magnetic interaction parameter $N \equiv \sigma B^2 L/\rho u$ that estimates the ratio between the Lorentz and inertia forces.

We focus on MHD turbulence far from the walls and consider the implications of the Lorentz force for large-eddy simulations. Modification of turbulence has been relatively thoroughly studied in analytical, experimental, and numerical works (see, for example, Schumann (1976), Alemany et al (1979), Zikanov and Thess (1998), or Vorobev et al (2005)). It has been understood that in unbounded flows the direct effect of the magnetic field is two-fold. First, there is an additional turbulence suppression via Joule dissipation of induced electric currents. Second, the flow acquires axisymmetric anisotropy with flow structures elongated in the direction of the magnetic field. The nature of the anisotropy becomes transparent if we use the Fourier transform to write the rate of the Joule dissipation of a mode $\widehat{\mathbf{u}}(\mathbf{k},t)$ as

$$\mu(\mathbf{k}) = \sigma B^2\rho^{-1}|\widehat{\mathbf{u}}(\mathbf{k},t)|^2\cos^2\theta, \tag{4}$$

where θ is the angle between the wavenumber vector $\mathbf{k}$ and the magnetic field $\mathbf{B}$. The Joule dissipation tends to eliminate the velocity gradients along the magnetic field lines, thus leading to elongation of flow structures. The flow approaches two-dimensional form with zero parallel gradients at $N \to \infty$, although it has been argued by Thess and Zikanov (2007) that the proper term is 'quasi-two-dimensionality' due to the inevitable elliptic and shear flow instabilities of the two-dimensional structures. Furthermore, pure two-dimensionality is impossible in the presence of walls non-parallel to the magnetic field (see, e.g. Sommeria and Moreau (1982)).

Only the anisotropy of gradients is directly affected by the magnetic field. Another type of anisotropy referring to inequality between the velocity components (anisotropy of the Reynolds stress tensor) can follow from the action of the magnetic field indirectly, through the nonlinear interaction mechanism. Other indirect effects include the suppression of the nonlinear energy transfer between the length scales and the associated increase of the inertial range slope of

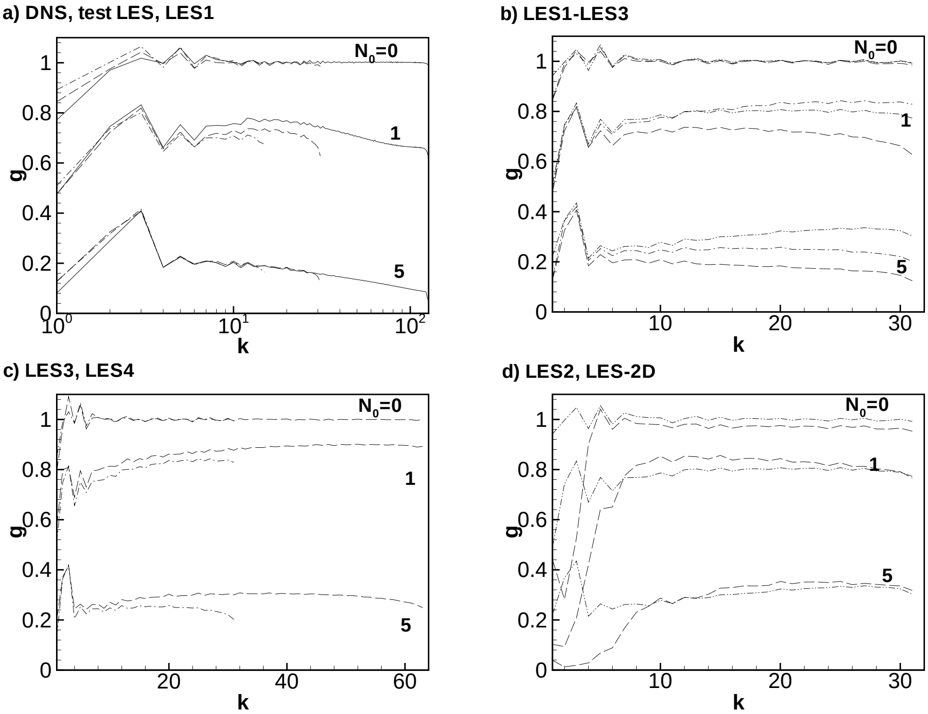

Figure 1: Results of Vorobev et al (2005). Gradient anisotropy coefficient (5) is shown as a function of k. *(a)*, DNS (——) and test LES at different filer widths ($- \cdot - \cdot -$) and (— — —) at $Re_\lambda = 92$. *(b)*, LES at $Re_\lambda = 92$, at $Re_\lambda = 140$ (— — —), and at $Re_\lambda = 290$ ($- \cdot - \cdot -$). *(c)*, LES at $Re_\lambda = 290$ with different filter widths. *(d)* LES at $Re_\lambda = 140$, with isotropic ($- \cdot\cdot - \cdot\cdot -$) and two-dimensional (— — —) forcing.

the energy power spectrum (approaching k^{-3} in quasi-two-dimensional flows at high N).

HOMOGENEOUS MHD TURBULENCE: RESULTS OF EARLIER STUDIES

A detailed study of the anisotropy of homogeneous MHD turbulence at low R_m was conducted by Vorobev et al (2005). DNS and LES computations were performed in a wide range of Re and N. In order to achieve a statistically steady flow, the artificial force was applied to the large scale modes with $1.5 \le k \le 3.1$. Two types of forcing were used, one isotropic with the energy input equally divided among the forced modes and another purely two-dimensional with the energy input limited to the modes with $\mathbf{k} \perp \mathbf{B}$.

The main results are illustrated in figure 1, which shows the scale-dependent anisotropy of flow gradients estimated as

$$g(k) \equiv \frac{3\tau}{2}\frac{\mu(k)}{E(k)} = \frac{3\sum \frac{k_z^2}{k^2}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}^*}{\sum \hat{\mathbf{u}}\cdot\hat{\mathbf{u}}^*}, \qquad (5)$$

where the sums are over all wavenumber vectors in the shell $k-1/2 < |\mathbf{k}| \le k+1/2$, and $\mu(k)$, $E(k)$ are the power spectra of the magnetic dissipation rate and kinetic energy. The scaling factor in (5) uses the Joule damping time $\tau \equiv \rho/\sigma B^2$ and is chosen so that $g(k) = 1$ in an isotropic flow and $g(k) = 0$ in a purely two-dimensional flow with zero magnetic dissipation.

It has been found that at all k outside the range of artificial forcing, $g(k)$ is a remarkably robust function of the magnetic interaction parameter N. The influence of the length scale, Re, and the details of the large-scale dynamics (dictated by the different types of the forcing) is much weaker. It has also been noticed that the value of $g(k)$ at small and moderate scales is well approximated by the global anisotropy coefficients

$$G_{ij} = \frac{\left\langle (\partial u_i/\partial z)^2 \right\rangle (1+\delta_{i3})}{\left\langle (\partial u_i/\partial x_j)^2 \right\rangle (1+\delta_{ij})},\ i=1,2,3,\ j=1,2, \qquad (6)$$

where $\langle \ldots \rangle$ stands for the volume averaging.

LES modeling of decaying and forced homogeneous MHD turbulence was conducted by Knaepen and Moin (2004) and Vorobev et al (2005), respectively. The Smagorinsky eddy viscosity closure hypothesis was used in both cases with the subgrid-scale stress tensor $\overline{\tau}_{ij}$ expressed through the rate-of-strain tensor of filtered velocity field $\overline{S}_{ij}$ as

$$\overline{\tau}_{ij} = -2C_S\overline{\Delta}^2|\overline{S}|\overline{S}_{ij}, \quad |\overline{S}| = \left(2\overline{S}_{ij}\overline{S}_{ij}\right)^{1/2}, \qquad (7)$$

where $\overline{\Delta}$ is the filter width and C_S is the Smagorinsky constant. The conclusion of both studies, achieved through the a-posteriori comparison with the results of high resolution DNS was that the Smagorinsky model is overdissipative if used in its simple form with constant C_S. Substantially better agreement with the DNS results was achieved when the constant C_S was determined using the dynamic procedure (Germano 1991, Lilly 1992).

Poor performance of the simple Smagorinsky model with constant C_S in the case of MHD flows with high N could be anticipated. The values used for C_S (between 0.01 and 0.0324 depending on the type and width of the filter and the strength of mean shear do not take into account the flow transformation caused by the applied magnetic field (anisotropy, suppressed nonlinear energy transfer, and steepened energy spectrum). All these factors lead to reduction of the subgrid-scale energy dissipation and, thus, require reduction of C_S. The better accuracy of the dynamic model is,

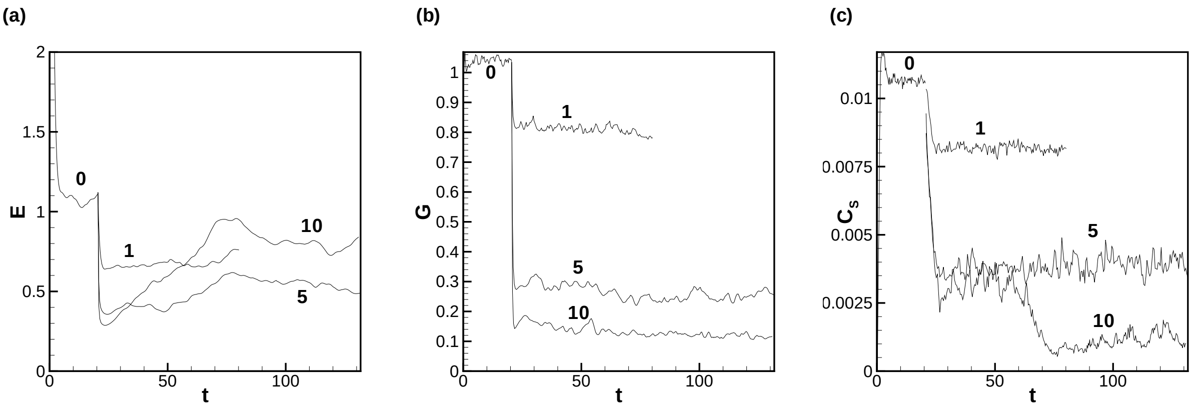

Figure 2: Flows with 3D forcing, $Re = 2500$, and numerical resolution $64 \times 64 \times 128$. Time evolutions of the resolved total energy *(a)*, global anisotropy coefficient *(b)*, and volume-averaged Smagorinsky coefficient *(c)* are shown for $N = 0$, 1, 5, and 10.

too, not entirely surprising. The model has demonstrated its ability to remedy the overdissipation problem in other situations characterized by reduced SGS dissipation, for example in the laminar-turbulent transition.

SMAGORINSKY CONSTANT AS A FUNCTION OF ANISOTROPY PARAMETERS

For our purposes, the discussion of the previous section can be reduced to the following two hypotheses, which, although not fully proven, received substantial factual support in LES of Knaepen and Moin (2004) and Vorobev et al (2005).

1. The dynamic adjustment of C_S is sufficient to account for anisotropy and other aspects of the flow transformation caused by the imposed magnetic field. No other modification of the Smagorinsky closure is needed to keep the accuracy of the model at the same level as in the isotropic case.

2. The variation of C_S caused by the magnetic field can be adequately described by a function of a single scalar parameter, such as the anisotropy coefficient (6) or the magnetic interaction parameter N.

Our goal is to capitalize on the hypotheses and obtain the relations $C_S = C_S(N)$ and $C_S = C_S(G)$. A brief discussion of the results is provided below. A more detailed description can be found in Vorobev and Zikanov (2007).

One can speculate on which of the two dependencies, $C_S(N)$ or $C_S(G)$, is preferable. There is no strong preference in the case of homogeneous turbulence, where, as shown by Vorobev et al (2005) and confirmed by our recent computations, G can approximately be considered a one-to-one function of N. In the real flows, where N is evaluated based on the size of the flow domain and some constant characteristic velocity, its effective value can change in space (for example because of variation of $\mathbf{B}$) or time (for example in the case of decaying flow). The relation $C_S = C_S(G)$ seems, therefore, preferable as the one based on a universal anisotropy characteristic that can be evaluated locally, both in space and time.

We present the results of a series of LES computations of homogeneous MHD turbulence in a box of dimensions $2\pi \times 2\pi \times 4\pi$ with periodic boundary conditions. The dynamic Smagorinsky model is used as a subgrid-scale closure. The numerical method is pseudo-spectral based on the fully dealiased Fast Fourier Transform. The flow is artificially forced at $1.5 \leq k \leq 3.1$ in the same manner as it was done by Vorobev et al (2005). Each numerical experiment starts with a non-magnetic run that lasts long enough to produce a fully developed turbulent flow. Then, at $t = t_0$, the magnetic field $\mathbf{B}$ is applied and kept constant till the flow transformation is complete, after which the flow statistics are collected and averaged over several (at least, 3) eddy turnover times $T = L(t_0)/u(t_0)$. The Reynolds number Re and the magnetic interaction parameter N are evaluated in the isotropic flow at $t = t_0$ using the integral length scale L and the rms velocity u. The experiments are conducted at $0 \leq N \leq 10$ and $Re = 700$, 2500, and 6500. Two numerical resolutions with $64\times64\times128$ and $32\times32\times64$ spectral functions are used. The filtering width $\overline{\Delta}$ of the LES closure is defined as the grid step. Further details of the computational procedure can be found in Vorobev et al (2005).

The typical temporal evolution of the flows is illustrated in figure 2. After the introduction of the magnetic field, the global characteristics such as the total resolved energy, rate of resolved viscous dissipation or the anisotropy coefficient G evolve rapidly until they stabilize at new levels corresponding to the forced anisotropic flows. The Smagorinsky constant evolves in the same way but, as can be seen from comparison between the figures 2a,b and 2c, its evolution to new levels is slower.

The flow anisotropy was estimated using the time-averaged global coefficient G and the scale-dependent coefficient $g(k_{\max})$ taken at the length scale of the filter width. One can see in figures 3a,b that both coefficients decrease rapidly at small and moderate N and somewhat slower at stronger fields with $N > 3$. The flow becomes strongly anisotropic but remains essentially three-dimensional. An important feature of curves in figures 3a,b is their closeness to each other. This is quite remarkable considering the fact that the curves are obtained for substantially different Reynolds numbers, forcing mechanisms, and numerical resolutions. Moreover, as can be seen in figure 3c, G is nearly equal to $g(k_{\max})$ and, thus, to coefficients $g(k)$ at any other k outside the range of forced length scales. Our calculations confirm the conclusions obtained by Vorobev et al (2005) on the basis of the DNS and LES computations at lower Reynolds numbers.

The main results are presented in figure 4, which shows the dependence of the volume- and time-averaged Smagorin-

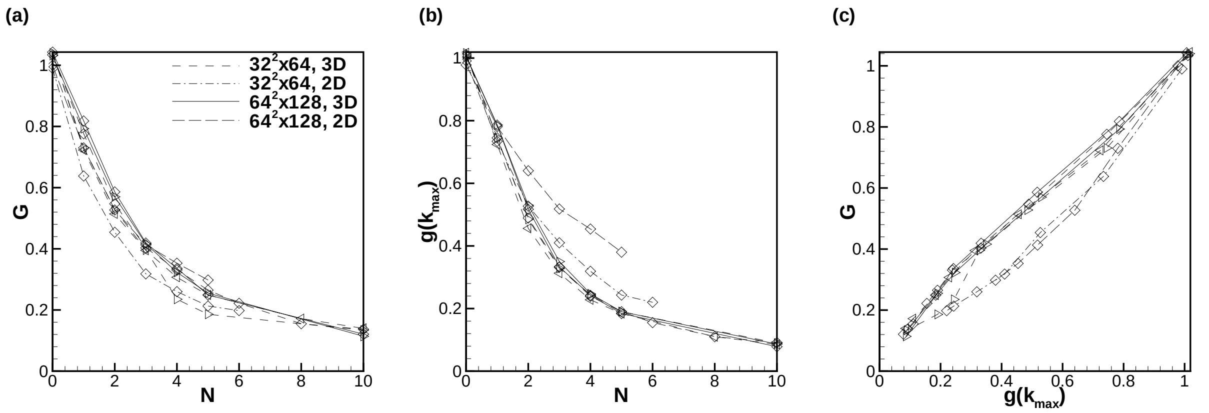

Figure 3: Global anisotropy coefficient *(a)* and the scale-dependent anisotropy coefficient taken at the scale of filter width *(b)* as functions of magnetic interaction parameter; *(c)*, Global vs. filter-width coefficients. $\triangleright$, $Re = 700$, $\diamond$, $Re = 2500$; $\triangleleft$, $Re = 6500$.

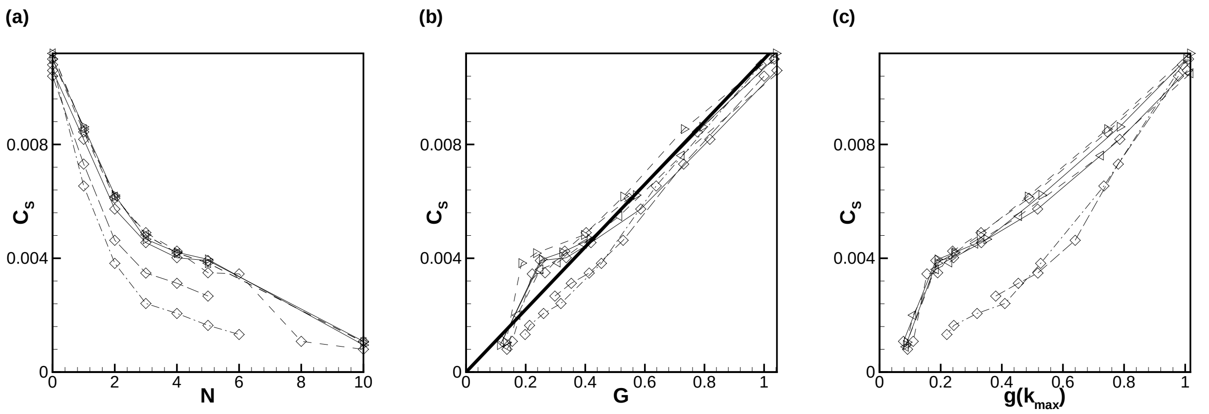

Figure 4: (a) Volume- and time-averaged Smagorinsky constant C_S as a function of the magnetic interaction parameter *(a)*, global anisotropy coefficient *(b)* and coefficient of anisotropy at filter width *(c)*. Bold line in *(b)* is for the linear relation (8). Notations are as in figure 3.

sky constant C_S on the anisotropy characteristics. As in the previous figures, data obtained at different Reynolds numbers, forcing mechanisms, and numerical resolutions are plotted. The constant decreases with the strength of the magnetic field as represented by increasing N or decreasing anisotropy coefficients. The results support the second of our hypotheses. The curves in figure 4 are quite close to each other. The Smagorinsky constant can be considered a function of N, G, or $g(k_{\max})$ with a reasonable degree of accuracy. One can see in figure 4b that the agreement is particularly good for the function $C_S = C_S(G)$. Requiring that C_S attains its isotropic value at $G = 1$ and zero in a purely two-dimensional flow at $G = 0$ we can approximate the data in figure 4b by a simple linear relation

$$C_S = C_{S0} G \tag{8}$$

shown by the bold line. Here C_{S0} is the Smagorinsky constant corresponding to the isotropic non-magnetic flow. It is about 0.011 in our calculations but can have different values in other cases depending on type of the flow and type and width of the filter.

In order to test the formula (8) we conducted simple numerical experiments. Forced flows with $Re = 2500$ and numerical resolution $64 \times 64 \times 128$ were computed using the dynamic Smagorinsky model, simple Smagorinsky model with constant $C_S = C_{S0}$, and the modification of the Smagorinsky model with constant C_S adjusted at each time step according to (8). Each run started with the same isotropic initial velocity field. Calculations were performed for $N = 1$ and $N = 5$. After completion of transitional periods, the statistics were collected and time-averaged for the fully established anisotropic flows.

The results are presented in figure 5. One can see that the adjustment (8) results in almost exact reproduction of the spectra of energy and dissipation rates of flows obtained with the dynamic Smagorinsky model. On the contrary, the spectra calculated with the Smagorinsky model with constant C_S are noticeably, albeit not very strongly, different, with typical overdissipative suppression at large k.

It must be stressed that such an excellent agreement between the dynamic model and the model with adjusted C_S is observed only for statistically steady periods of the flow evolution. The situation during the transient periods is quite different as illustrated by our test simulations of decaying turbulence. In the simulations, a fully developed forced non-magnetic flow is used as an initial conditions. At the moment $t = t_0$, the magnetic field is applied, and the forcing is discontinued, after which the flow is allowed to decay freely for several turnover times simultaneously developing anisotropy. Figure 6 shows the results of the dynamic LES of the process for $N = 1$ and $N = 5$. One can see in figures 6a,b that both G, which represents C_S/C_{S0} determined according to (8),

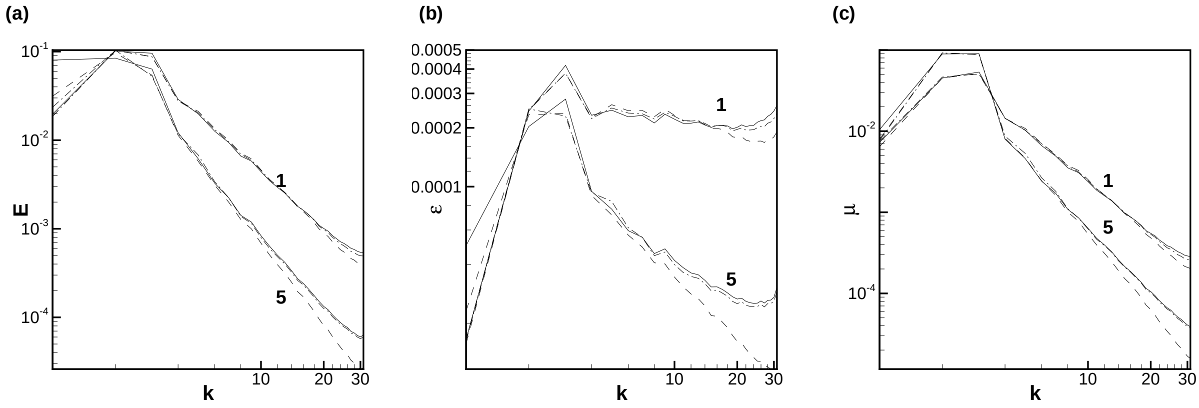

Figure 5: Spectra of resolved energy *(a)*, viscous *(b)*, and magnetic *(c)* dissipation rates. Flows with 3D forcing, $Re = 2500$, $N = 1$ and 5, and numerical resolution by $64 \times 64 \times 128$ functions are calculated using the dynamic model (——), Smagorinsky model with constant C_S (– – –), and Smagorinsky model with constant C_S adjusted according to (8) (– · – · –).

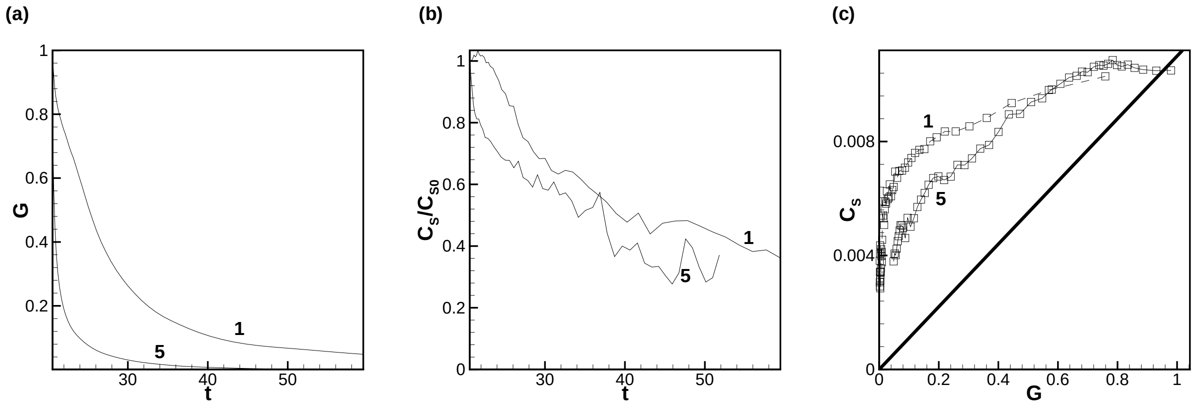

Figure 6: Decaying turbulent flows with $Re = 2500$, $N = 1$ and 5 calculated using the dynamic model. *(a)*, global anisotropy coefficient G, which also represents C_S/C_{S0} calculated according to the linear relation (8); *(b)*, Smagorinsky constant C_S scaled by the isotropic value C_{S0}; *(c)*, C_S vs. G

and the dynamically determined C_S/C_{S0} decrease with time as the flow becomes anisotropic but at very different rates. Initially, the decay rate for the dynamic C_S is almost zero, whereas the anisotropy coefficient drops rapidly. At later stages, C_S decays faster than G but remains significantly larger than predicted by (8). This is further illustrated in figure 6c. Similar delay in the modification of the Smagorinsky constant is demonstrated by the forced flows during the transitional stages of their development (see figures 2b,c).

One can be tempted by a simple explanation that the generation of dimensional anisotropy is an essentially linear process governed by the Joule dissipation (4). Its typical time scale is the Joule damping time $\tau \equiv \rho/\sigma B^2$. On the other hand, the variation of the Smagorinsky constant is associated with the nonlinear process of establishing new correlations between the turbulent rate of strain and stress tensors occurring at the time scale of the eddy turnover time $T = L/u$. One can expect slower evolution of C_S at $N = T/\tau > 1$. The explanation, albeit possibly relevant, is clearly an oversimplification. In particular, it does not explain the delay in the development of C_S at $N = 1$ (see figure 6).

CONCLUDING REMARKS

The main result of our study is a confirmation of the hypothesis that the dynamic adjustment of the Smagorinsky constant in the case of magnetically suppressed turbulence can be accurately approximated by a simple linear function of the global coefficient of gradient anisotropy. Apart from purely theoretical interest, the relation has a potential practical value. Calculation of C_S in the dynamic model, if done at the every time step, approximately doubles the amount of computations in comparison with the standard Smagorinsky model. This is undesirable in simulations of industrial processes such as, for example, the Czochralski growth of large crystals or continuous steel casting, the tasks computationally challenging even with the simplest turbulence models. With quantified dependence of C_S on G one could still use the standard model but avoid its overdissipative character by adjusting C_S to the strength of the flow anisotropy.

We found that the adjustment formula cannot be used for computation of essentially transient processes because the development of anisotropy occurs at a faster rate than the modification of the Smagorinsky constant. Furthermore, the formula was obtained for homogeneous turbulence and may prove inaccurate in the presence of mean shear or rotation. It is possible that improved correlations can be developed for such situations but they are likely to be also more complex and less universal than our simple formula. Their advantage over the dynamic model is, thus, far from being obvious.

Acknowledgements

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