ON SELF-SMILARITY OF WALL-BOUNDED FLOWS

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ABSTRACT

It is still unknown if in general wall-bounded turbulence shows self-similarity in the sense that a set of scaling variables exists to collapse the profiles of, for example, all moments of the streamwise velocity component. To find self-similar solutions for the mean-velocity profile and the Reynolds shear stress, usually a scaling approach is employed where the characteristic length and velocity scales are defined *a priori*. Herein we undertake the challenge of non-dimensionalizing the mean-momentum equation of turbulent boundary layers without such *a priori* specification.

INTRODUCTION

Self-similarity is a long and intensively researched aspect of wall-bounded flows. One of the main objectives of these investigations is to collapse data obtained in different facilities (e.g., wind and water tunnels), or under different physical circumstances (e.g., different pressure gradients or different wall roughness), within a single curve. The practical relevance of self-similar solutions is clear. For example, the well-known classical logarithmic law having constant parameters is employed in nearly all commercial Navier–Stokes solvers as embedded wall function. However, if this law would fail, these numerical schemes would fail too (Gad-el-Hak, 1997).

Most approaches to find self-similar solutions of wall-bounded flow search for coordinate transformations of the governing equations (e.g., Prandtl's boundary layer equation, mean-momentum equation of pipe and channel flows, etc.). Clauser (1956) introduced the fundamental idea of equilibrium or self-preservation. The solutions he sought exhibit self-similarity, meaning that the governing equations do not show any explicit dependence on the streamwise coordinate. Therefore, the goal is to normalize those equations in a way that the mean-velocity profile, Reynolds shear-stress profile, etc., collapse for different flow realizations.

Early solutions of this type were provided by Rotta (1950), Clauser (1956) and Michel et al. (1969), among others. During the last decade, the problem of finding self-similar solutions received renewed interest. New approaches were developed by Barenblatt et al. (2000), Castillo and George (2001), and Buschmann and Gad-el-Hak (2003), among others. However, despite the large number of solutions developed for turbulent wall-bounded flows, none of the scaling approaches advanced thus far seems to be completely satisfactory. A most recent overview on the scaling approaches developed during the last decade is provided by Buschmann and Gad-el-Hak (2007).

PRESENT APPROACHES

The majority of approaches attempting to find self-similar solutions start with an assumption for the scaling of the mean-velocity profile and the Reynolds shear-stress. The list below provides some examples.

Classical approach (Schlichting and Gersten, 1997)

$$\eta = y / \delta(x) \tag{1a}$$

$$f_{cl}'(\eta) = \frac{u_e(x) - u(x,y)}{u_\tau(x)}\ ; \quad r_{cl}(\eta) = -\frac{\langle u'v' \rangle (x,y)}{\left[u_\tau(x)\right]^2} \tag{1b, c}$$

Alternative approach (Wolfshtein, 2004)

$$\eta = y / \delta(x) \tag{2a}$$

$$f_{al}'(\eta) = \frac{u(x,y)}{u_e(x)}\ ; \qquad r_{al}(\eta) = -\frac{\langle u'v' \rangle (x,y)}{\left[u_e(x)\right]^2} \tag{2b, c}$$

Castillo–George scaling (Castillo and George, 2001)

$$\eta = y / \delta(x) \tag{3a}$$

$$f_{op}(\eta) = \frac{u(x,y) - u_e(x)}{u_{So}(x)}\ ; \quad r_{So}(\eta) = -\frac{\langle u'v' \rangle (x,y)}{R_{So}(x)} \tag{3b, c}$$

Zagarola–Smits scaling (Zagarola and Smits, 1997)

$$\eta = y / \delta(x) \tag{4a}$$

$$f_{ZS}'(\eta) = \frac{u(x,y)}{u_e(x)} \frac{1}{\delta^*(x)/\delta(x)} \tag{4b}$$

$$r_{ZS}(\eta) = -\frac{\langle u'v' \rangle (x,y)}{R_{ZS}(x)} \tag{4c}$$

Here x denotes the streamwise and y the wall-normal coordinate, δ is the boundary layer thickness, and δ^* the displacement thickness. The velocity in mean-flow direction is u, and the Reynolds shear-stress is $\langle u'v' \rangle$. The variable u_e denotes the freestream velocity.

It is still unknown if in general wall-bounded turbulence shows self-similarity in the sense that a set of scaling variables exists to collapse all moments of the streamwise velocity. Certain experiments rather indicate that this is not the case (e.g., Morrison et al., 2004). Therefore, the present discussion will be restricted to the profiles of the mean-velocity and the Reynolds shear-stress, as the majority of all approaches do too.

The present scaling approaches [Equations (1–4)] have a similar structure—all of them start with the non-dimensionalization of the wall-normal coordinate, the mean-velocity profile and the Reynolds shear-stress. The scaling variables assumed are different, however. While the classical approach starts with $u_\tau(x)$, the GC-scaling (George–Castillo scaling) leaves the determination of the characteristic velocity open until several constraints from the analyzed boundary-layer equations are derived (Castillo and George, 2001). Similarly the scaling of the Reynolds shear-stress is classically assumed to be $\left[u_\tau(x)\right]^2$, but found to be $u_e^2(x)\, d\delta / dx$ according to GC-scaling by employing the constraints following from the analyzed governing equations.

For the ZS-scaling (Zagarola–Smits scaling), $u_e(x)\, \delta^*(x) / \delta(x)$ is used for scaling the mean-velocity profile. This scaling was originally based on empirical ground by Zagarola and Smits (1997) but later shown to be an extension of the classical two-layer approach (Panton, 2005; Indinger et al., 2006).

Obviously the outcome of a certain approach depends not only on the physical assumptions made but also on the mathematical (functional) structure allowed *a priori*. The assumptions made in the approaches shown in Equations (1–4) already pre-justify the results possible to obtain. Herein, we instead propose a more general approach of scaling with lesser assumptions concerning the functional shape of the scaling and the scaling variables.

UNSPECIFIED APPROACHE

We consider a turbulent boundary layer subjected to a pressure gradient. In the new approach we follow in principle the procedure described above but without specifying the characteristic velocity and length scales employed

$$\eta = f_1(x)\, f_2(y) \tag{5a}$$

$$f'(\eta) = \frac{u(x,y)}{u_*(x)}; \qquad r(\eta) = -\frac{\langle u'v' \rangle (x,y)}{R(x)} \tag{5b, c}$$

The functions $f_1(x)$ and $f_2(y)$, the scaling velocity $u_*(x)$ and the scaling of the Reynolds shear-stress $R(x)$ are not specified *a priori*. This permits maximum flexibility with respect to the scaling variables, includes all approaches listed in the previous section, and allows—after incorporating into the mean-momentum balance—a general discussion of that equation.

Next we consider the two-dimensional, steady, incompressible boundary layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - u_e \frac{d u_e}{d x} = \frac{\partial}{\partial y}\left(\nu \frac{\partial u}{\partial y} - \langle u'v' \rangle \right) \tag{6}$$

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

is employed to substitute the wall-normal velocity component in (6). The *ansatz* (5) is then introduced into (6), and after a considerable amount of algebra the dimensionless mean-momentum equation is obtained in the following form

$$\frac{u_*^x}{\nu} \frac{1}{f_1^2} \left(f'^2 - f\, f'' - \frac{u_e^x}{u_*^x} \frac{u_e}{u_*} \right) + \underbrace{\frac{u_*}{\nu} \frac{f_1^x}{f_1^3}}_{Term\,B} f\, f'' = \underbrace{\frac{f_2^{yy}}{f_1}}_{Term\;D} f'' + \left(f_2^y \right)^2 f''' + \frac{R}{\nu} \frac{1}{u_*} \frac{f_2^y}{f_1} r' \tag{8}$$

Here the prime denotes derivative with respect to η, the superscripts x and y symbolize, respectively, the derivative with respect to the x- and y-coordinate.

One of the primary goals of non-dimensionalization of (6) is to get rid of one coordinate. In principle, this transforms the partial differential Equation (6) into an ordinary differential equation. This can only be achieved in (8) if $f_2(y)$ is set identical to y. In that case, the first derivative of $f_2(y)$ becomes identical to unity, and *Term D* that contains the second derivative of $f_2(y)$ becomes identically zero.

APPROACHE ACCORDING TO WOLFSHTEIN

We employ the approach originally derived by Wolfshtein (2004) to find self-similar solutions for wall-bounded flows with injection to illustrate the classical way of finding self-similar solutions. Insert Equations (2a–c) into (8) to yield

$$\underbrace{\frac{\delta^2 u_*^{x}}{\nu}}_{Term\ A_{al}}\left(f_{al}'^2 - f_{al}\, f_{al}''-1\right) - \underbrace{\frac{\delta\,\delta^x u_e}{\nu}}_{Term\ B_{al}} f_{al}\, f_{al}'' = f_{al}''' + \underbrace{\frac{\delta\, u_e}{\nu}}_{\mathrm{Re}\, ynolds\, number} r_{al}' \quad (9)$$

For laminar flow, the shear-stress term r_{al}' becomes zero. Self-similarity is therefore obtained if the two remaining similarity numbers A_{al} and B_{al} are constant. For turbulent flow, the shear-stress term is non-zero and the additional constraint $\mathrm{Re} = u_e\delta/\nu = const.$ has to be satisfied.

By differentiating the Reynolds number and substituting the resulting velocity gradient into A_{al}, one obtains

$$A_{al} = -B_{al} = \mathrm{Re}\,\delta^{x} = const. \quad (10)$$

which finally reduces (9) to

$$A_{al}\left(f_{al}'^2 - 1\right) = f_{cl}''' + \mathrm{Re}\, r_{al}' \quad (11)$$

It is immediately clear that with this approach any self-similar solution must have two parameters. To satisfy similarity both Re and A_{al} have to be constant. The parameter A_{al} is related to the Clauser–Rotta parameter β, sometimes called outer pressure-gradient parameter

$$\beta = \frac{\delta^*}{\rho u_\tau^2}\frac{d\,p}{d\,x} \quad (12)$$

$$A_{al} = -\beta\frac{\delta}{\delta^*}\,\mathrm{Re}\frac{c_f}{2} \quad (13)$$

The straightforward consequence from the above analysis is that a turbulent boundary layer having zero-pressure-gradient is not strictly speaking a self-similar or equilibrium flow. The only self-preserving boundary layer on a smooth surface is therefore the sink flow. This flow has constant Clauser–Rotta parameter and constant local Reynolds number, meaning that the ratio of pressure and friction forces and the ratio of inertia and friction forces are both constant along the development length of the boundary layer.

GC-APPROACHE

To show that Equation (8) encompasses the approach by Castillo and George (2001), we specify (5a–c) according to (3a–c) and obtain Castillo and George's version of the mean-momentum equation written in outer variables

$$\left[\delta\frac{u_e^{x}}{u_{so}} + \frac{\delta}{u_{so}}\frac{u_e}{u_{so}}\frac{d\,u_{so}}{d\,x}\right] f_{op\infty} + \left[\delta\frac{u_{so}^{x}}{u_{so}}\right] f_{op\infty}^2 - \left[\delta^{x} + \delta\frac{u_e^{x}}{u_{so}}\right]\eta\, f'_{op\infty}$$

$$-\left[\delta^{x} + \delta\,\frac{u_{so}^{x}}{u_{so}}\right] f'_{op\infty}\int_0^{\eta_{CG}} f_{op\infty}\, d\,\eta = \left[\frac{\mathrm{Re}_{so}}{u_{so}^2}\right] r'_{op\infty} \quad (14)$$

In the above, the similarity numbers appear in square brackets. However, differently from the classical view, however, those numbers do not have to be constant. To obtain full similarity of the mean momentum Castillo and George (2001) demand only that these terms have to show the same *x* dependence. This so-called GC-equilibrium-type similarity is fundamentally different from the definition of equilibrium flow according to Clauser (1956).

For turbulent boundary layers having zero-pressure gradient, the last term on the left-hand side of (14) becomes identically zero. In this case, only two constraints follow from (14), otherwise more constraints have to be enforced. For example, the outer region of TBL with pressure gradient follows

$$\Lambda = -\frac{\delta}{u_e}\frac{d\,u_e / d\,x}{d\,\delta / d\,x} \quad (15)$$

According to the definition by Castillo and George (2001), equilibrium flow occurs when the parameter Λ becomes constant.

ZS-APPROACHE

In a third attempt, we employ the scaling according to Zagarola and Smits (1997). With

$$u_*(\,x\,) = u_{ZS}(\,x) = u_e(x)\,\frac{\delta^*(x)}{\delta(x)} \quad (16)$$

the normalization of the mean-velocity profile is given. The normalization for the Reynolds shear-stress is not known *a priori*. We therefore employ the coefficient of the shear-stress term in (8), the last term in the equation. Knowing that this should be a Reynolds number [see also last term of Equation (9)], we equate

$$\frac{u_e\,\delta_*}{\nu} = \frac{R_{ZS}}{\nu}\frac{1}{u_{ZS}}\frac{f_2^{\,y}}{f_1} \quad (17)$$

with $f_2^{\,y} = 1$ and $f_1 = 1/\delta$, and obtain

$$R_{ZS} = u_{ZS}^2 = u_e^2\,\frac{\delta_*^2}{\delta^2} \quad (18)$$

Note that differently from the usual scaling applying u_τ^2, the Reynolds shear-stress is scaled with a combination of the boundary layer thickness, displacement thickness and the velocity at the outer edge of the boundary layer. That of course supports the idea that freely moving large scales influence the Reynolds shear-stress over the largest part of the boundary layer (see, e. g., Morrison et al., 2004). That the scaling proposed with (18) is similarly good as the usual

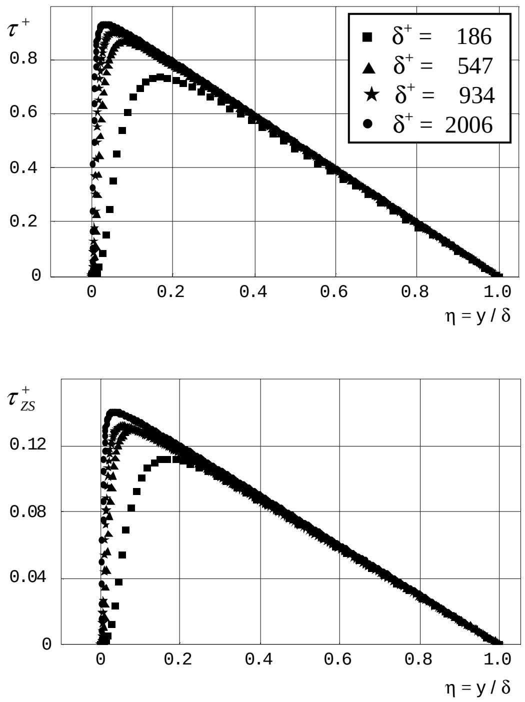

Figure 1: Distribution of Reynolds shear stress of channel flow (DNS-data from Hoyas Jiménez, 2005)
Above usual scaling based on u_τ and below scaling according to eq. (18)

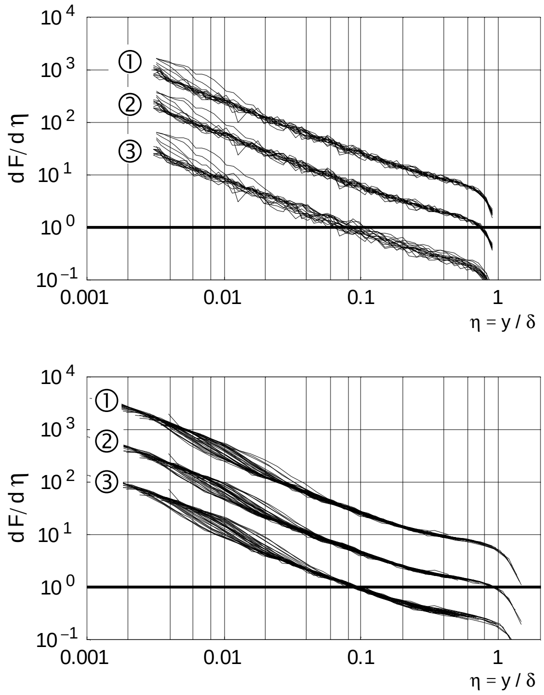

Figure 2: Gradient of the defect law for different scaling approaches for pipe flow
Above superpipe data from McKeon et al. (2004) and below turbulent boundary layer with zero pressure gradient (Österlund, 1999)

① Classical approach $u_* = u_\tau$

② Zagarola–Smits scaling $u_* = u_{ZS}$

③ Castillo-George scaling $u_* = u_e$

scaling using u_τ^2 shows Figure 1 for channel flow DNS data from Hoyas and Jimenez (2005).

By specifying *Term B* of eq. (8) according to (4a-c) it is found that

$$\frac{u_*}{\nu}\frac{f_1{}^x}{f_1{}^3} = -\frac{u_e\delta^*}{\nu}\delta^x \quad \text{and} \quad -\mathrm{Re}_{ZS}\,\delta^x = const. \tag{19}$$

This constraint is basically the same as the one found by Wolfshtein (2004) with eq. (10). This finding supports again the close relationship between the classical scaling and the ZS-scaling as discussed by Indinger et al. (2006). The conclusion is that self-similarity employing ZS-scaling can be obtained only if equivalent constraints are demanded, as for the classical outer scaling.
Substituting all terms of (8) according to the ZS-scaling finally leads to the mean-momentum equation in the form

$$-\mathrm{Re}_{ZS}\,\delta^x\left[f_{ZS}{}'^2\underbrace{-\frac{\delta^{*2}}{\delta^2}-\frac{u_e}{u_e{}^x}\left(\frac{\delta^*}{\delta}\frac{\delta^{*x}}{\delta}+\frac{\delta^{*2}}{\delta^2}\frac{\delta^{*x}}{\delta}\right)}_{Term\ K_{ZS}}\right] = f_{ZS}''' + \mathrm{Re}_{ZS}\,r_{ZS}' \tag{20}$$

To obtain self-similarity *Term* K_{ZS} must be constant. From this a second constraint for the ZS-scaling follows. The Reynolds number Re_{ZS} is written as

$$\mathrm{Re}_{ZS} = \frac{u_e\delta^*}{\nu} \quad \text{and} \quad u_e = \mathrm{Re}_{ZS}\frac{\nu}{\delta^*} \tag{21a, b}$$

For a flow having constant properties, the displacement thickness is inversely proportional to u_e. Taking the derivative of (21b) with respect to x leads to the following relation

$$1 = -\frac{u_e}{u_e^x}\frac{\delta^{*x}}{\delta^*} \tag{22}$$

Introducing (22) into *Term* K_{ZS} yields

$$K^{1/3}\delta = \delta^* \tag{23}$$

which basically demands that boundary layer thickness and displacement thickness have to develop proportionally along the streamwise direction. This implies a constant shape factor, δ^*/δ, which is a characteristic feature of classical self-similarity of TBL subjected to pressure gradient (Skåre and Krogstad, 1994).

To summarize, employing the unspecified normalized mean-momentum equation (8), it is found that certain constraints have to be fulfilled to achieve self-similarity using ZS-scaling. These constraints are equivalent to the constraints demanded for classical self-similarity. However, why does the ZS-scaling provide a better collapse of pipe flow data as compared to the classical approach?

Following the classical picture (Tennekes and Lumley, 1972), it is stated that well above the surface layer

$$\frac{h}{u_*}\frac{dU}{dy}=\frac{dF}{d\eta} \quad \text{with} \quad u_*=u_\tau \tag{24}$$

with the understanding that $dF/d\eta$ is of order unity. Calculating the gradient of the defect law from the superpipe data (McKeon et al., 2004) and from the ZPG TBL-data from Österlund (1999) shows that for the classical scaling ($u_*=u_\tau$) this is only true in the outermost region of the flow, as shown in Figure 2.

Doing the same employing Castillo–George scaling ($u_*=u_e$) shifts the curves below the unity-line and the gradient in the outer region is of order 0.1. The Zagarola–Smits ($u_*=u_{ZS}$) scaling comes closest to this line in a wide portion of the outer region, which may explain its practical success.

All three scalings are related because for infinite Reynolds numbers, the scaling velocities become proportional. However, the collapse obtained with the classical or the ZS-scaling is better in the outer region of the flow ($\eta \geq 0.1$), which supports the argument that u_e might be not be the proper velocity scale for the defect law.

CONCLUSION

In this paper we make an unspecified *ansatz* to non-dimensionalize the mean-momentum equation of turbulent boundary layers. An immediate outcome of this exercise is that the coordinate y can only occur linearly in the dimensionless wall-coordinate. Otherwise the mean-momentum equation does not reduce to an ordinary differential equation. Whether new classes of self-similar solutions can be found following this track remains open.

Because the characteristic length and velocity scales are not assumed *a priori* in the present formulation, the resulting dimensionless mean-momentum equation can be employed for the purpose of comparing different approaches. Employing the Zagarola–Smits scaling, it is shown that equivalent constraints as for the classical scaling have to be fulfilled to obtain self-similarity. Additionally, the adequate ZS-scaling for the Reynolds shear stress is derived from the dimensionless mean-momentum equation.

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