EVALUATION OF DIFFUSION/TRANSPORT CONSTRAINTS FOR ARSM IN CALCULATION OF FULLY DEVELOPED ROTATING CHANNEL FLOW

Jifeng Qiu, Shinnosuke Obi
Department of Mechanical Engineering, Keio University,
Hiyoshi, Kohoku-ku Yokohama 223-8522, Japan
y12477@educ.cc.keio.ac.jp, obsn@mech.keio.ac.jp

Thomas B. Gatski
Laboratoire d'Études Aérodynamiques
UMR 6609 CNRS/Université de Poitiers/ENSMA,
BP 30179, 86962 Futuroscope Chasseneuil Cedex, France
Center for Coastal Physical Oceanography and Ocean, Earth and Atmospheric Sciences,
Old Dominion University, Norfolk, VA 23529, USA
t.b.gatski@widomaker.com

ABSTRACT

The weak-equilibrium condition, which is the basis in the development of algebraic Reynolds stress models, is assessed in the case of fully developed turbulent channel flow under the influence of system rotation. The budget of the various terms in the exact transport equation for the anisotropy tensor is evaluated using a DNS database. An asymptotic analysis of the near-wall behavior is performed and an alternative form of the diffusion/transport constraint is proposed. The analysis shows that the proposed alternative diffusion/transport constraint has the potential to improve the predictive ability of the ARSM.

INTRODUCTION

There continues to be considerable interest in the development of algebraic Reynolds stress models (ARSM) in order to meet the increasing demands in industrial applications. The general algebraic relation is obtained by a simplification of the differential transport equations of Reynolds stress anisotropies. This results in implicit algebraic relations for the Reynolds stress components. An explicit form of the ARSM (EARSM) is obtained from a tensor polynomial representation of the anisotropy components whose terms are functions of mean strain rates and vorticity tensors as well as additional scalar parameters. The EARSM thus inherits the simplicity and some level of robustness of the eddy viscosity models but also retains the potential for representing the turbulence anisotropy.

The classic weak-equilibrium condition (Rodi, 1972, 1976) is used to derive the implicit ARSM from the differential transport equation of Reynolds stress anisotropy. For the mean convection term, this condition gives the first assumption that the advection of the Reynolds stress anisotropy tensor in turbulent flows is zero,

$$\frac{Db_{ij}}{Dt} = 0 \tag{1}$$

where $b_{ij} = \tau_{ij}/2k - \delta_{ij}/3$ and $\tau_{ij} = \overline{u_i u_j}$. For the turbulent transport and viscous diffusion, this same condition yields the second assumption that the turbulent transport and viscous diffusion of τ_{ij} is proportional to that of the turbulent kinetic energy k, that is

$$\mathcal{D}_{ij} - \frac{\tau_{ij}}{k}\mathcal{D} = 0 \tag{2}$$

where $\mathcal{D}_{ij}$ is the turbulent transport and viscous diffusion of the Reynolds stress given by

$$\begin{aligned}\mathcal{D}_{ij} =& \mathcal{D}^t_{ij} + \mathcal{D}^p_{ij} + \mathcal{D}^\nu_{ij} \\ =& -\frac{\partial}{\partial x_k}\overline{u_i u_j u_k} \\ & -\frac{\partial}{\partial x_k}\left(\overline{pu_i}\delta_{jk} + \overline{pu_j}\delta_{ik}\right) + \nu\nabla^2\tau_{ij}\end{aligned} \tag{3}$$

where $\mathcal{D}^t_{ij}$ is the turbulent transport, $\mathcal{D}^p_{ij}$ is the pressure transport and $\mathcal{D}^\nu_{ij}$ is the viscous diffusion ($\mathcal{D} = \mathcal{D}_{ii}/2$). The present study focuses on the validity of the diffusion/transport constraint expressed in Eq. (2).

The assumption (Eq. (2)) is based on a constraint applicable to the turbulent transport and viscous diffusion (Gatski and Rumsey, 2002), and is necessary in deriving the implicit algebraic Reynolds stress equations. There has not been much attention paid to this assumption since it becomes important only in near-wall regions of inhomogeneous turbulent flows and is less amenable to analysis than the condition on the Reynolds stress anisotropy itself. However, with the aid of DNS data, it is possible to analyze the near-wall behavior of the individual terms in the transport equation of anisotropy tensor.

In the present study, the diffusion/transport constraint problem is explored by means of budget and near-wall asymptotic analysis. An alternative form for the diffusion/transport constraint is then proposed. The *a priori* test of both original and proposed diffusion/transport constraints is performed, and it is demonstrated that the proposed form is able to improve the predictive ability of ARSM.

BUDGET OF ANISOTROPY TENSOR EQUATION

The exact transport equation of b_{ij} for fully developed rotating channel flow is given by

$$
\begin{aligned}
0 = & \left(P_{ij} + C_{ij} - \frac{\tau_{ij}}{k} P\right) + \phi_{ij} \\
& - \left(\varepsilon_{ij} - \frac{\tau_{ij}}{k} \varepsilon\right) + \left(\mathcal{D}_{ij} - \frac{\tau_{ij}}{k} \mathcal{D}\right) \qquad (4)
\end{aligned}
$$

where

$$
\begin{aligned}
P_{ij} =& - \tau_{ik} \frac{\partial \overline{u_j}}{\partial x_k} - \tau_{jk} \frac{\partial \overline{u_i}}{\partial x_k} && (5a) \\
C_{ij} =& - 4k \left(e_{imk} \Omega_m b_{kj} + e_{jmk} \Omega_m b_{ki}\right) && (5b) \\
\phi_{ij} =& \overline{\frac{p}{\rho} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} && (5c) \\
\varepsilon_{ij} =& 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} && (5d)
\end{aligned}
$$

The terms on the RHS of Eq. (4) are analogous to measures of the anisotropies associated with the production, redistribution, dissipation and diffusion/transport.

The turbulent kinetic energy equation can be obtained from a contraction of indices in Eq. (4) as

$$
0 = P - \varepsilon + \mathcal{D} \qquad (6)
$$

where the terms on the RHS represent the turbulent production $P = -\tau_{ik} \partial \overline{u_i} / \partial x_k$, the isotropic turbulent dissipation rate $\varepsilon = \varepsilon_{ii}/2$ and the combined effects of turbulent transport and viscous diffusion $\mathcal{D}$.

The budget between the various terms in Eq. (4) is evaluated by using a DNS database (Kristoffersen and Andersson, 1993). Figure 1 shows the budget of the b_{ij} equations for $Ro = 0.0$. For the b_{11}-component, Figure 1(a) shows that the production anisotropy term is the dominant source, while the redistribution term is the dominant sink. As the wall is approached, the production anisotropy and redistribution terms decay rapidly, while the diffusion/transport and dissipation anisotropy terms are the dominant source and sink - this is a viscous effect due to the close proximity to the wall.

For the b_{22}-component, the behavior of the individual terms in the budget equation is significantly different from those in the b_{11}-equation. Since this component is strongly influenced by the wall reflection effect, the near-wall behavior is affected by the pressure fluctuations. Figure 1(b) shows that the dominant source in this case is the redistribution term, while the term associated with the production anisotropy acts as a sink. As the wall is approached, the production and dissipation anisotropy terms become less important, while the redistribution term and diffusion/transport anisotropy term - which are affected by the pressure fluctuations, are in balance and reach finite values on the wall.

Figure 1(c) shows that for the b_{33}-component, the term associated with the production anisotropy is the dominant sink in contrast to the b_{11} equation, while the redistribution term is the dominant source in the center of the channel. As the wall is approached, the production anisotropy and the redistribution terms decay rapidly. In the near-wall region, the viscous terms, i.e., the diffusion/transport anisotropy and dissipation anisotropy, are dominant.

In the equation for the b_{12}-component, Figure 1(d) shows that the redistribution term is the dominant source term throughout the channel. The production anisotropy term is the dominant sink in the center of the channel, but it decays rapidly in the near-wall region and vanishes at the wall. The diffusion/transport anisotropy becomes significant near the wall and keeps balance with the redistribution term.

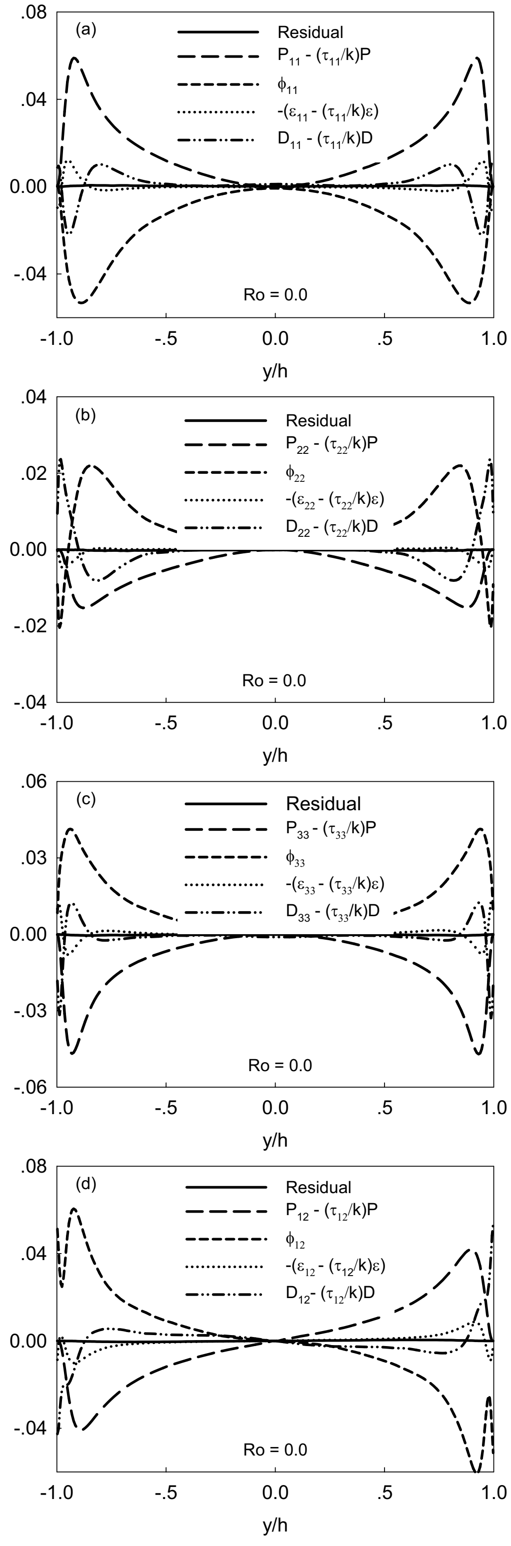

Figure 1: The budget of Eq. (4) for b_{ij} at $Ro = 0$

Probably due to the local isotropy, the dissipation anisotropy is less important for this component throughout the channel.

For the rotating cases, the significant influence of rotation can be observed by examining Figure 2, where the turbulent intensity is enhanced along the pressure side, while reduced along the suction side; however, some common features with the non-rotating case exist. The production anisotropy is small in the near-wall region for all four

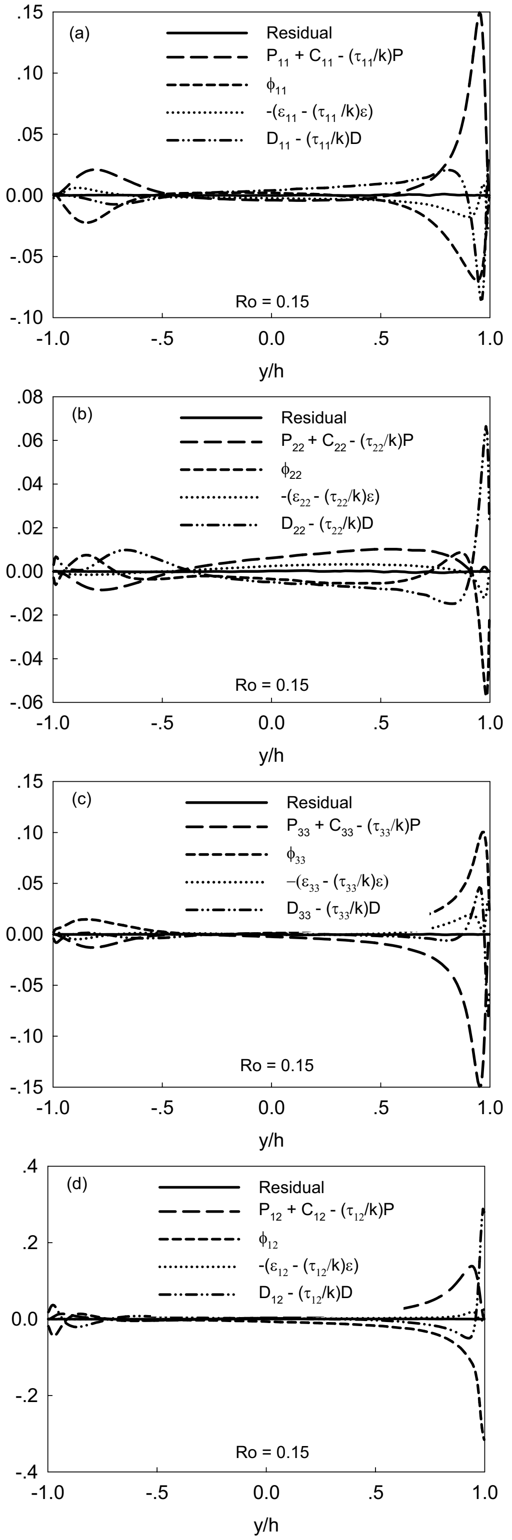

Figure 2: The budget of Eq. (4) for b_{ij} at $Ro = 0.15$

components, and the diffusion/transport and dissipation anisotropy are dominant for the b_{11}- and b_{33}-components in the wall vicinity. On the other hand, the diffusion/transport anisotropy and redistribution are dominant there for the b_{22}- and b_{12}-components.

For all the cases examined, it appears that the redistribution term balances the sum of the diffusion/transport anisotropy and dissipation anisotropy. This means that the diffusion/transport anisotropy plays a crucial role in the b_{ij} transport equation in the near-wall region, and also suggests that the diffusion/transport constraint, Eq. (2), is unlikely to hold in the near-wall region. In the center of channel, the absolute levels of the residuals are not negligibly small. Although this suggests that neglect of the diffusion/transport anisotropy term is also unlikely to hold for the center of the channel, the present study will focus on the diffusion/transport constraint in the near-wall region.

MODIFICATION TO DIFFUSION/TRANSPORT CONSTRAINT

The analysis in the preceding section has shown that the diffusion/transport anisotropy is not negligible in the budget of the b_{ij} equation, suggesting the inadequacy of the current diffusion/transport constraint. The discussion now will focus on a possible modification to the diffusion/transport constraint. First, the near-wall behavior of the individual terms in the budget equation of the anisotropy tensor is examined.

Near-wall behavior of b_{ij} equation

Figure 3 shows the budget of b_{11} equation in the vicinity of the wall. For the $Ro = 0$ case, the production is the dominant source in the range $y^+ \geq 10$, while the redistribution is the dominant sink. As the wall is approached, the production anisotropy and redistribution terms decay rapidly, while the dissipation and diffusion/transport anisotropy terms remain finite and balance each other to the wall. For the $Ro = 0.15$ and $Ro = 0.50$ cases, it is readily observed that the effects of rotation have a significant influence on the budget, but the wall proximity behavior remains unchanged, e.g., the rapid decay of the production anisotropy term, and a finite value for the diffusion/transport and dissipation anisotropy terms.

For the b_{12} equation, the asymptotic near-wall behavior of the terms in the Eq. (4) is now investigated in more detail. The velocity and pressure variables in the vicinity of the wall can be expanded in wall units as (Patel et al., 1985; Mansour et al., 1988; So et al., 1997)

$$u^+ = b_1 y^+ + \cdots \tag{7a}$$
$$v^+ = c_2 y^{+2} + \cdots \tag{7b}$$
$$w^+ = b_3 y^+ + \cdots \tag{7c}$$
$$p^+ = a_p + b_p y^+ + \cdots \tag{7d}$$

The wall-limiting behavior of the individual terms in b_{12}-equation are expressed in the same manner so that

$$P_{12} + C_{12} - \frac{\overline{uv}}{k}P = -2\Omega_3^+ \overline{b_1^2} y^{+2} + \cdots \tag{8a}$$
$$\phi_{12} = \overline{a_p b_1} + \cdots \tag{8b}$$
$$\varepsilon_{12} - \frac{\overline{uv}}{k}\varepsilon = 2\overline{b_1 c_2} y^+ + \cdots \tag{8c}$$
$$\mathcal{D}_{12} - \frac{\overline{uv}}{k}\mathcal{D} = -\overline{a_p b_1} + \cdots \tag{8d}$$

Figure 4 shows the budget of the b_{12} equation. For the non-rotating and rotating cases, the dominant source is the production an isotropy term, while the dominant sink is the redistribution term throughout most of the channel. As the wall is approached, the dissipation anisotropy decays as $\mathcal{O}(y^+)$, and the production anisotropy decays faster as $\mathcal{O}(y^{+2})$. The redistribution term remains relatively large, and keeps balance with the diffusion/transport anisotropy

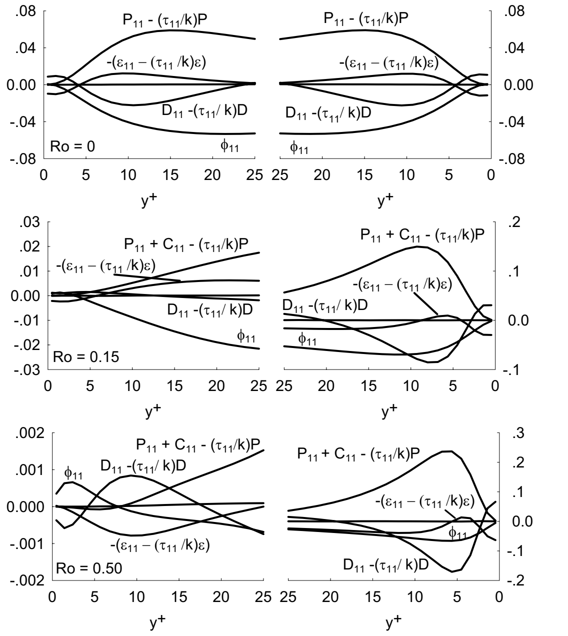

Figure 3: Terms in the budget of b_{11} in wall coordinates (*lhs*: suction side, *rhs*: pressure side)

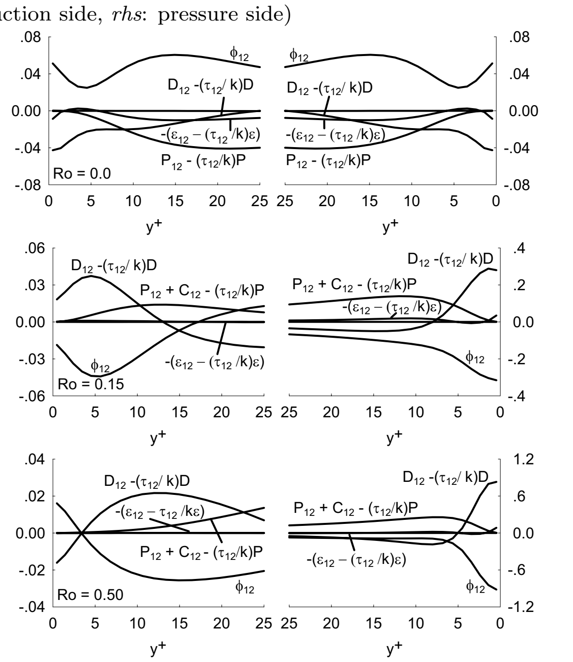

Figure 4: Terms in the budget of b_{12} in wall coordinates (*lhs*: suction side, *rhs*: pressure side)

term for $y^+ \leq 10$. The above expansions (Eq. (8)) also indicate that in the vicinity of the wall, the redistribution and diffusion/transport anisotropy terms decay as $\mathcal{O}(y^0)$. At the wall, the redistribution balances with the diffusion/transport.

In the RANS modeling framework, the velocity pressure-gradient Π_{ij} is usually split into the pressure transport $\mathcal{D}^p_{ij}$ and the redistribution ϕ_{ij}. The expansions of these three

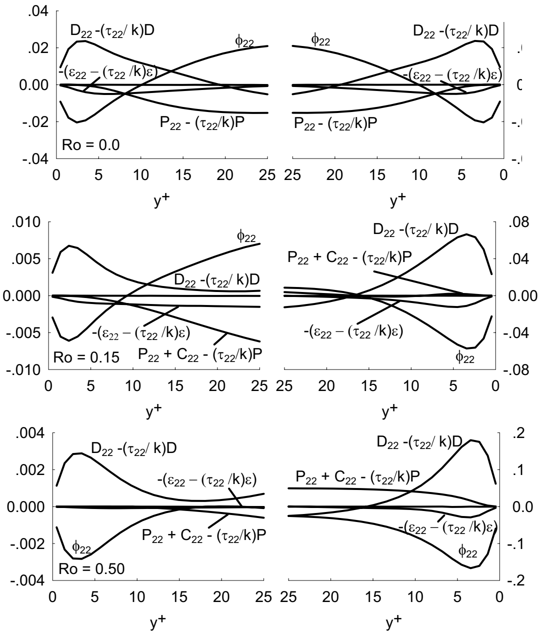

Figure 5: Terms in the budget of b_{22} in wall coordinates (*lhs*: suction side, *rhs*: pressure side)

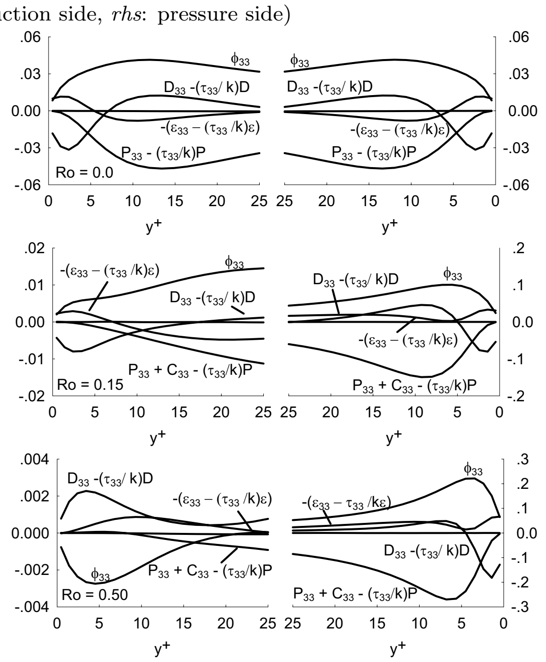

Figure 6: Terms in the budget of b_{33} in wall coordinates (*lhs*: suction side, *rhs*: pressure side)

terms for b_{12}-component are

$$\Pi_{12} = -2\overline{b_1 c_2} y^+ + \cdots \tag{9a}$$

$$\phi_{12} = \overline{a_p b_1} + 2\left(\overline{b_1 c_2} + \overline{a_p c_1}\right) y^+ + \cdots \tag{9b}$$

$$\mathcal{D}^p_{12} = -\overline{a_p b_1} - 2\left(2\overline{b_1 c_2} + \overline{a_p c_1}\right) y^+ + \cdots \tag{9c}$$

It is shown that Π_{12} is of $\mathcal{O}(y^+)$, while ϕ_{12} and $\mathcal{D}^p_{12}$ are of $\mathcal{O}(y^0)$, thus Π_{12} is negligible in the near-wall region compared to ϕ_{12} and $\mathcal{D}^p_{12}$. It is clear that ϕ_{12} must balance with $\mathcal{D}^p_{12}$ in the near-wall region. Because $\mathcal{D}^p_{12}$ is the major contributor to the diffusion/transport $\mathcal{D}_{ij} - (\tau_{ij}/k)\mathcal{D}$, cf. Eqs. (8d) and (9c), it is fair to conclude that the diffusion/transport anisotropy must balance with the redistribution in the near-wall region and cannot be neglected.

Figure 5 shows the budget of the b_{22} equation. The redistribution becomes negative at about $y^+ \approx 10$, while the diffusion/transport anisotropy becomes large and remains in balances with the redistribution for $y^+ \leq 10$ in a manner similar to the b_{12} case. Both Figure 5 and the near-wall expansions show that production anisotropy decays as $\mathcal{O}(y^{+3})$, and the dissipation anisotropy as $\mathcal{O}(y^{+2})$. If the velocity pressure-gradient partitioning is examined in the same manner as for b_{12}- component, Π_{22} decays as $\mathcal{O}(y^{+2})$, and becomes negligible for $y^+ \leq 10$. Thus, the diffusion/transport anisotropy also balances the redistribution in the near-wall region. Figure 6 shows the budget of the b_{33} equation. The overall tendency is similar to the other components, except for the production anisotropy term that is now significant close to the wall on the pressure side. Analogue to the case in b_{11}- component, the dissipation anisotropy and the diffusion/transport anisotropy have finite values on the wall and remain in balance with each other.

Near-wall correction of diffusion/transport constraint

The above analysis of the wall asymptotic behavior of the terms in the b_{ij} equations may be summarized as follows: in the b_{11}- and b_{33}-equations, viscous diffusion dominates so that the whole diffusion/transport process balances with the viscous dissipation in close proximity to the wall; in the b_{12}- and b_{22}-equations, pressure transport and redistribution are the major contributors in close proximity to the wall. The near-wall modeling strategy of the diffusion/transport term must cope with these two different mechanisms, and this is readily accomplished by adopting the form

$$\mathcal{D}_{ij} - \frac{\tau_{ij}}{k}\mathcal{D} = -\left[\phi_{ij} - \left(\varepsilon_{ij} - \frac{\tau_{ij}}{k}\varepsilon\right)\right] f_d \qquad (10)$$

with f_d being a function that restricts the effect of the redistribution term and dissipation within the near-wall region. The form of f_d is specified so that it becomes unity on the wall, and gradually decays away from the wall for $y^+ \geq 10$. Initially, the following form is proposed

$$f_d = 1 - \left[1 - \exp\left(\frac{y^+}{6}\right)\right]^2 . \qquad (11)$$

It is adopted here for its simplicity, but the development of a more general form should be a task for future study.

EVALUATION OF PROPOSED CONSTRAINT

In order to evaluate the original and proposed diffusion/transport constraints, an *a priori* test is performed using the DNS data of Kristoffersen and Andersson (1993). Applying the original diffusion/transport constraint to Eq. (4), one obtains

$$0 = \frac{1}{2k}\left(P_{ij} + C_{ij} - \frac{\tau_{ij}}{k}P\right) + \frac{\phi_{ij}}{2k} - \frac{1}{2k}\left(\varepsilon_{ij} - \frac{\tau_{ij}}{k}\varepsilon\right) \quad (12)$$

By adopting the proposed diffusion/transport constraint, Eq. (12) can now be written as

$$\begin{aligned} 0 =& \frac{1}{2k}\left(P_{ij} + C_{ij} - \frac{\tau_{ij}}{k}P\right) \\ &+ \left[\frac{\phi_{ij}}{2k} - \frac{1}{2k}\left(\varepsilon_{ij} - \frac{\tau_{ij}}{k}\varepsilon\right)\right](1 - f_d) \end{aligned} \qquad (13)$$

This equation shows that the proposed diffusion/transport constraint has the effect of removing both the redistribution and dissipation anisotropy terms near the wall. It should be noted that the validity of this form is unaffected by the system rotation since the balance between the redistribution and pressure transport, and between the viscous diffusion and dissipation persist regardless of the rotation number (cf. Figures 3 – 6). In another analysis of the non-rotating and rotating channel flow cases, Manceau (2005) also showed that the asymptotic behavior in the near-wall region was unaffected.

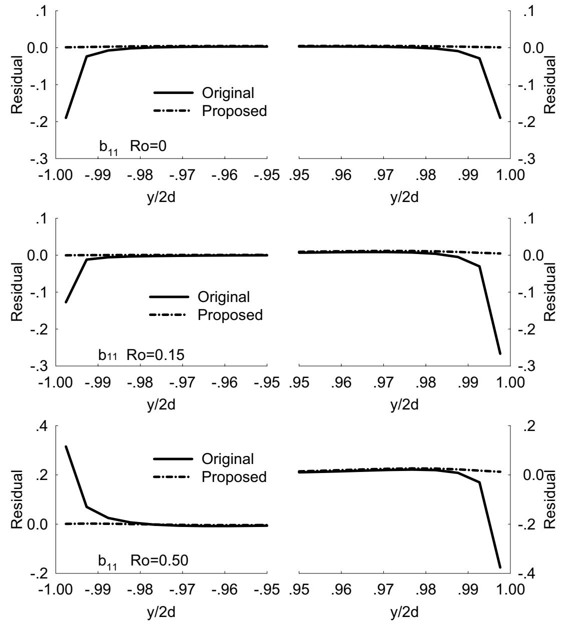

Figure 7: Validation of present diffusion/transport constraint for b_{11}-component (*lhs*: suction side, *rhs*: pressure side)

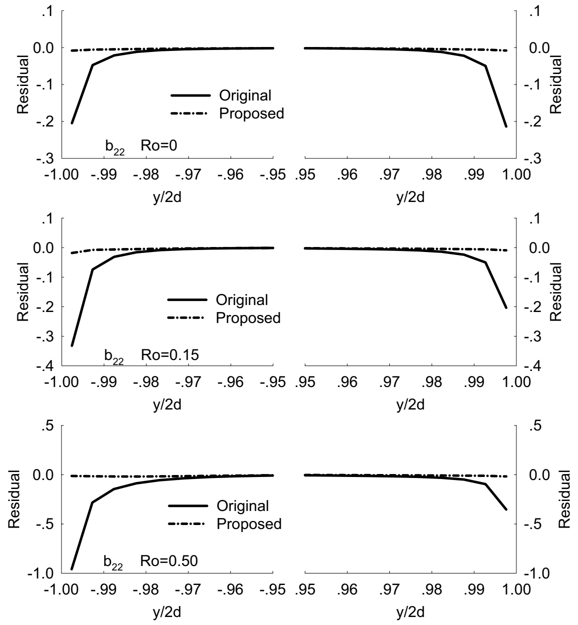

Figure 8: Validation of present diffusion/transport constraint for b_{22}-component (*lhs*: suction side, *rhs*: pressure side)

Figure 7 shows the results for b_{11}-component, where the proposed diffusion/transport constraint gives nearly zero residual for the b_{11}-component in the vicinity of the wall. For the other components, Figures 8 – 10 show that the

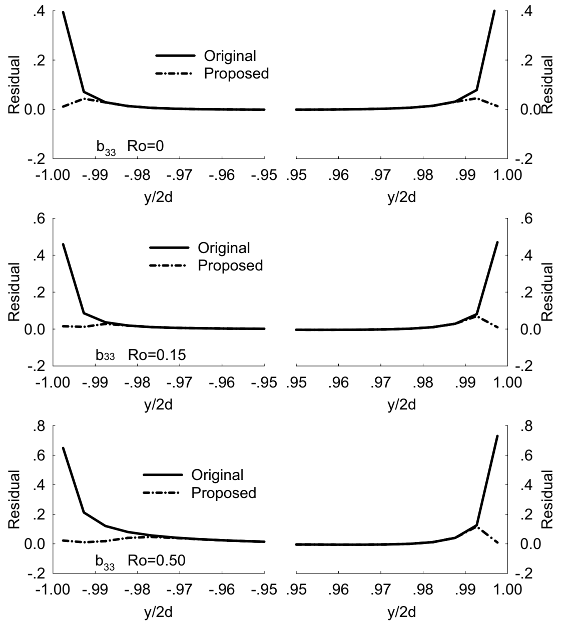

Figure 9: Validation of present diffusion/transport constraint for b_{33}-component (*lhs*: suction side, *rhs*: pressure side)

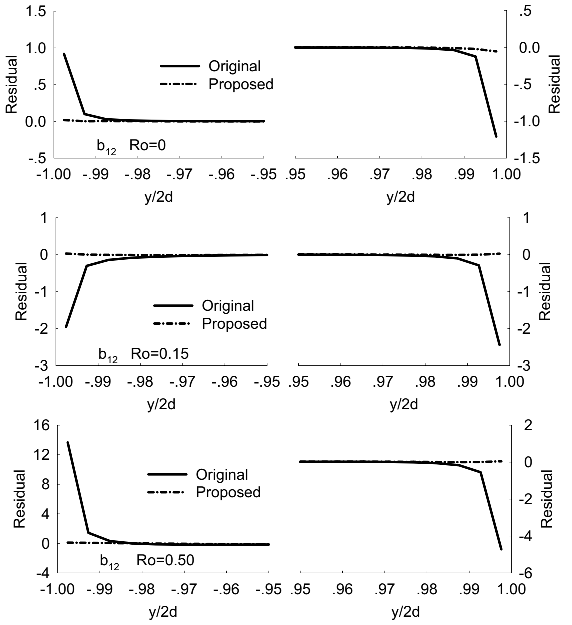

Figure 10: Validation of present diffusion/transport constraint for b_{12}-component (*lhs*: suction side, *rhs*: pressure side)

present diffusion/transport constraint can also significantly reduce the residuals compared to the original formulation in Eq. (12). Thus, there is an obvious improvement compared with the original form for both the non-rotating and rotating cases.

There is, however, a slight imbalance observed in Figure 9 on the pressure side for the b_{33}-component. This is attributable to the increase of the production anisotropy term there (cf. Figure 6), which adversely affects the previous mentioned balance between the diffusion/transport anisotropy term and its counterparts. A more advanced modeling strategy is necessary to cope with this issue.

Nevertheless, the above analysis shows that proposed alternative diffusion/transport constraint has the potential to improve the predictive capabilities of the ARSM once accurate models for the redistribution and dissipation rate terms in Eq. (4) are provided.

CONCLUSION

The budget analysis of the various terms in the exact transport equation for b_{ij} show that the diffusion/transport anisotropy term is crucial in the near-wall region. An asymptotic analysis of the near-wall behavior shows that the diffusion/transport anisotropy term keeps balance with the sum of the redistribution and dissipation anisotropy term in the vicinity of the wall, while the production anisotropy is small. An alternative form of the diffusion/transport constraint is proposed and evaluated using DNS data. Evaluation results show that proposed alternative diffusion/transport constraint has the potential to improve the predictive capability of the resultant ARSM.

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