A LES-LANGEVIN MODEL FOR TURBULENCE

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ABSTRACT

We propose a new model of turbulence for use in large-eddy simulations (LES). The turbulent force, represented here by the turbulent Lamb vector, is divided in two contributions. The contribution including only subfilter fields is deterministically modeled through a classical eddy-viscosity. The other contribution including both filtered and subfilter scales is dynamically computed as solution of a generalized (stochastic) Langevin equation between periodic reinitializations. This equation is derived using Rapid Distortion Theory (RDT) applied to the subfilter scales. The general friction operator therefore includes both advection and stretching by the resolved scale. The stochastic noise is derived as the contribution from the energy cascade described by the aliasing. The LES model is thus made of an equation for the resolved scale, including the turbulent force, and a generalized Langevin equation integrated on a twice-finer grid. The model is validated by comparison to DNS and is tested against classical LES models for isotropic homogeneous turbulence, based on eddy viscosity.

INTRODUCTION

The need of Large Eddy Simulation (LES) modelisation is rooted in our inability to handle all the degrees of freedom of a large Reynolds number turbulent flow. The general strategy of LES is to try to exactly handle the largest scales of the flow while parameterizing the global effect of the subgrid scales. Most of the LES models are designed to correctly reproduce the dissipative part of the turbulent stress tensor via a turbulent viscosity tuned so as to stabilize the flow. However, the effect of subgrid scales is generally much more complex than a simple dissipation. An important challenge for modeling is indeed to reproduce the complete energy transfer from subgrid scales to resolved scales, a phenomenon sometimes referred to as "backscatter". Pure viscosity models cannot reproduce this phenomenon, even through negative viscosity. Detailed investigations of the interactions between resolved and subgrid scales have shown that a large part of the turbulent tensor is due to interactions with scales down to half the cutoff scales (Domaradzki et al., 1994). This behavior motivates the introduction of new type of models based on the estimation of the largest subgrid scales to try to capture the back-scatter. The estimated scales can be either a random process obeying given statistics, or can be inferred from an analysis of the nonlinear terms of resolved scales (Domaradzki and Saiki, 1997). Another class of models compute the subgrid scales via approximate equations. These multi-levels models can not be considered as LES models if the cost of the integration of small scales is too important. Dubois *et al* proposed a dynamic multi-level model based on the hypothesis of approximated manifolds (Dubois and Jauberteau, 1998). However, the model implies as many degree of freedom as a Direct Numerical Simulations (DNS).
Our strategy relies upon the modeling of the subgrid scales by a stochastic process defined through an appropriate dynamic Langevin equation. The model is based upon Rapid Distortion Theory (RDT). Our strategy to use an additional approximate equation for the largest subgrid scales is close to recent subgrid-scale estimation models advocated by Domaradzki and his collaborators (Domaradzki and Saiki, 1997) . One of the main difference is the choice of the subgrid scales equation. In our model, we derived an equation for the cross stress tensor from the Navier Stokes equation using the RDT hypothesis. The basic hypothesis of non-locality in scales has been studied theoretically and experimentally for different flows (Carlier et al., 2001, Dubrulle et al., 2002, Laval et al., 2003, Laval et al., 2001). The model is derived independently of the filter and does not require a deconvolution of the filtered quantities. A similar model has already been successfully validated in two-dimensional turbulence (Laval et al., 1999, Laval et al., 2004).

In this paper, we derive the Langevin equation of the subgrid scales and test the model by comparison with DNS. Then we explain how this model can be used in the LES context by projection onto the near-cutoff subgrid scales. The resulting model will be compared to several other existing LES models for isotropic homogeneous turbulence.

DERIVATION OF THE MODEL

Consider a turbulent flow, with velocity field $\mathbf{u}(\mathbf{x}, t)$ and introduce a filtering procedure so as to separate it into a large-scale field $\overline{\mathbf{u}}$ and a subgrid field $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$. The large-scale field obeys a dynamical equation obtained by filtering of the Navier-Stokes equation, which may conveniently be written as (Wu et al., 1999):

$$\partial_t \overline{\mathbf{u}} + \overline{(\overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}})}_\perp + \overline{\boldsymbol{\ell}}_\perp + \overline{(\boldsymbol{\omega}' \times \mathbf{u}')}_\perp = \nu \Delta \overline{\mathbf{u}} \qquad (1)$$

$$\boldsymbol{\ell} = \overline{\boldsymbol{\omega}} \times \mathbf{u}' + \boldsymbol{\omega}' \times \overline{\mathbf{u}}. \quad (2)$$

Here, $\boldsymbol{\omega}'$ and $\overline{\boldsymbol{\omega}}$ are the subgrid and large scale vorticity and ν is the viscosity. The subscript $\perp$ means divergence-free component. The interaction between resolved and subgrid scales mainly responsible, for the energy backscatter are singled out in the quantity $\boldsymbol{\ell}$. Using an RDT approximation the equation for $\mathbf{u}'$ can be written as:

$$\partial_t \mathbf{u}' + \boldsymbol{\ell}'_\perp = (\nu + \nu_t^\star)\Delta \mathbf{u}' - \mathbf{f}'_\perp. \quad (3)$$

where the non-linear interactions between subgrid scales are modeled by a dissipative term $\nu_t^\star \Delta \mathbf{u}'$. The quantity $\mathbf{f}'_\perp = (\overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}})_\perp - \overline{(\overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}})}_\perp$ is a forcing stemming from the energy cascade. The equivalent equation for the subgrid vorticity $\boldsymbol{\omega}'$ is obtained by taking the curl of eq. (3). Combining the equation for $\boldsymbol{\omega}'$ with the equation for $\mathbf{u}'$, and using the fact that subgrid scales vary over fast time scale with respect to large scale we can write the equation for $\boldsymbol{\ell}$:

$$\begin{aligned} \partial_t \boldsymbol{\ell} &\approx \overline{\boldsymbol{\omega}} \times \partial_t \mathbf{u}' + \partial_t \boldsymbol{\omega}' \times \overline{\mathbf{u}}, \\ &\approx -\left(\overline{\boldsymbol{\omega}} \times (\boldsymbol{\ell}'_\perp + \mathbf{f}'_\perp)\right) + \left(\nabla \times (\boldsymbol{\ell}'_\perp + \mathbf{f}'_\perp) \times \overline{\mathbf{u}}\right) \\ &\quad +(\nu + \nu_t^\star)\left(\overline{\boldsymbol{\omega}} \times \Delta \mathbf{u}' + \Delta \boldsymbol{\omega}' \times \overline{\mathbf{u}}\right). \quad (4) \end{aligned}$$

In order to obtained a closed equation for $\boldsymbol{\ell}$, the last two terms of eq. (4) are lumped into a dissipative term $(\nu + \nu_t^\star)\Delta \boldsymbol{\ell}$. For simplicity, we use a constant turbulent viscosity $\nu_t^\star$. In this model the nonlinear part corresponding to the subgrid stress tensor is modeled using a Langevin equation. In order to get a practical LES model, one needs to introduce a model for the term $\overline{(\boldsymbol{\omega}' \times \mathbf{u}')}_\perp$. This term is modeled by a simple dissipation term $\nu_t \Delta \overline{\mathbf{u}}$ where ν_t only depends on time and is linked to the energy at the cutoff scale. Finally we get the following RDT model:

$$\begin{aligned} \partial_t \overline{\mathbf{u}} &+ \overline{(\overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}})}_\perp + \overline{\boldsymbol{\ell}}_\perp = (\nu + \nu_t)\Delta \overline{\mathbf{u}} \quad (5) \\ \partial_t \boldsymbol{\ell}_\perp &+ \left\{\left(\overline{\boldsymbol{\omega}} \times (\boldsymbol{\ell}'_\perp + \mathbf{f}'_\perp)\right) + \left(\nabla \times (\boldsymbol{\ell}'_\perp + \mathbf{f}'_\perp) \times \overline{\mathbf{u}}\right)\right\}_\perp \\ &= (\nu + \nu_t^\star)\,\Delta \boldsymbol{\ell}_\perp \quad (6) \end{aligned}$$

THE PRACTICAL MODEL

In order to build a practical LES model, one needs to restrict the integration of the equation (6) to a limited number of degree of freedom. When the typical size Δ of the filter used to split large and small scales is within the inertial range as require for a LES simulation, the intensity of $\boldsymbol{\ell}$ is maximum near Δ. Therefore, the integration of the $\boldsymbol{\ell}$ equation on a grid twice as large than the grid used for resolved scales do not lower significantly the performance of the LES model. In this paper, all the validation of the model will be performed with a mesh ratio of two. Integrating only the largest scale of $\boldsymbol{\ell}$, the dissipative term $\nu_t^\star \Delta \boldsymbol{\ell}_\perp$ has been replaced by an hyper-dissipative term $\nu_t^\star \Delta^p \boldsymbol{\ell}_\perp$ in order to limit the dissipation to the smallest scales. The resulting model is adapted to any geometry and the only parameter is in the modelisation of the Reynolds stress tensor $\overline{(\boldsymbol{\omega}' \times \mathbf{u}')}_\perp$. However, in some case, the LES model may support additional approximations. In the spirit of quasi-linear approximation, the transport and the stretching by resolved scales as well as the dissipative term may be replaced by a friction term $-\boldsymbol{\ell}_\perp/\tau$ where τ is a typical time scale. The resulting model is:

$$\begin{aligned} \partial_t \overline{\mathbf{u}} &+ \overline{(\overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}})}_\perp + \overline{\boldsymbol{\ell}}_\perp = (\nu + \nu_t)\Delta \overline{\mathbf{u}} \quad (7) \\ \partial_t \boldsymbol{\ell}_\perp &= -\boldsymbol{\ell}_\perp/\tau - \left\{\overline{\boldsymbol{\omega}} \times \mathbf{f}'_\perp + (\nabla \times \mathbf{f}'_\perp) \times \overline{\mathbf{u}}\right\}_\perp \quad (8) \end{aligned}$$

The resulting model looks like a classical Langevin model, where the friction is provided by the transport and stretching by the large scale, and the stochastic forcing $\xi = -\left\{\overline{\boldsymbol{\omega}} \times \mathbf{f}'_\perp + (\nabla \times \mathbf{f}'_\perp) \times \overline{\mathbf{u}}\right\}_\perp$ originates from the energy cascade through the cut-off scale. In this respect, it is natural to define the friction time as a typical correlation time with respect to the large scales. The simplest choice is to link the correlation time with the gradient tensor $S = \nabla \cdot \overline{\mathbf{u}}$. In the following tests, we used $\tau = 2(S:S)^{1/2}$.

THE RESULTS

As a first validation step, the RDT model (eqs. 5-6) and the Langevin model (eqs. 7-8) have been tested numerically by comparison with high resolution DNS and other LES models for decaying and forced isotropic turbulence.

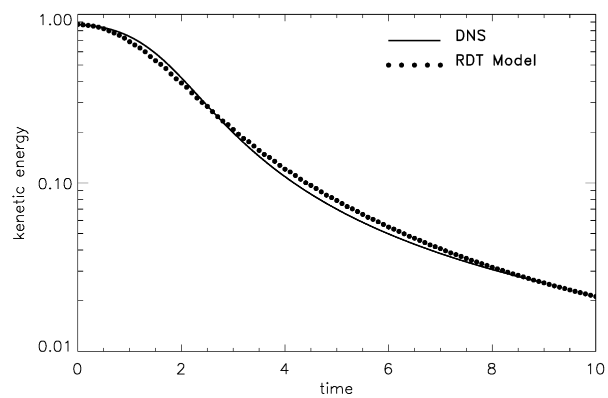

Figure 1: Comparison of evolution of the kinetic energy for our RDT model (eqs. 5-6) and the corresponding DNS of homogeneous isotropic turbulence. The LES with our model was performed with 85^3 effective Fourier modes for the equation of ℓ. A cutoff filter at $k_c = 21$ was used to separate resolved and subgrid scales. The DNS was performed with 342^3 effective (i.e. after dealiasing) Fourier modes. The simulations are initialized with a random velocity field with Gaussian statistics such as the initial kinetic energy spectrum is $E(k, t_o) = \alpha k^4 e^{-k^2/8}$. The Taylor Reynolds number varies from $R_\lambda = 260$ to $R_\lambda = 26$ during the simulation.

In Figure 1, the energy decay obtained with the RDT model (eqs. 5-6) is compared to the same quantity of the equivalent high resolution DNS. In this validation, the term $\overline{(\boldsymbol{\omega}' \times \mathbf{u}')}_\perp$ was modeled with a turbulent viscosity linked to the level of kinetic energy at the cut-off scales $\nu_t = C_{\mathbf{u}}\, k_c^{-1/2}\, E(k_c)^{1/2}$. The constant $C_{\mathbf{u}}$ was adjusted to get the best fit of energy spectra. This model is equivalent to the spectral model (Lesieur, 1990), but the constant needs to be lowered as the cross stress tensor is already modeled by $\boldsymbol{\ell}$. The results show that the modelisation of $\boldsymbol{\ell}$ by the Langevin equation (6) is accurate even if the additional turbulent viscosity $\nu_t^\star$ introduced in the equation of $\boldsymbol{\ell}$ to dump the smallest scales has not been optimized. In the numerical tests, we choose $\nu_t^\star = C_\ell k^2 k_c^{-1/2} E(k_c)^{1/2}$. The constant $C_\ell = 0.002$ was adjusted to get enough dissipation at the smallest scales in order stabilize the equation. Several values have been tested and the accuracy of the model is not very sensitive to this parameter.

In a second test, the simplified Langevin model (eqs. 7-8) has been compared to an equivalent high resolution DNS and two other LES of forced isotropic homogeneous turbulence. The averaged energy spectra are shown Fig. 2. The four simulations were initialized with the same velocity field (extracted from a previous DNS) and were integrated over 4 turnover times. Our Langevin model gives better results. Unlike the two other LES models, our model is able to reproduce the right $k^{-5/3}$ slope which means that the model leads to a correct dissipation near the smallest resolved scales.

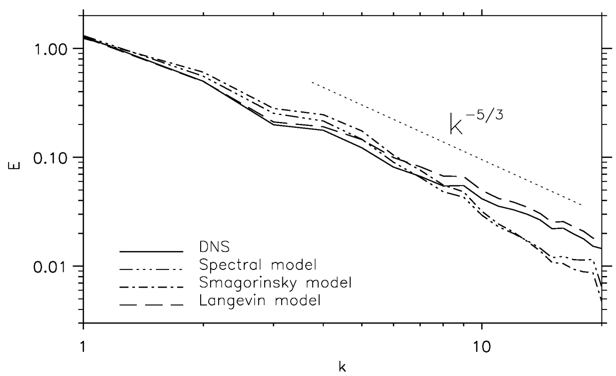

Figure 2: Comparison of the mean kinetic energy spectra for forced homogeneous isotropic turbulence. The DNS was performed with 342^3 effective (i.e. after dealiasing) Fourier modes, and the three LES were performed with 42^3 effective Fourier modes using a cutoff filter at $k_c = 21$. For our LES model (eqs. 7-8) the equation of ℓ was integrated with 85^3 effective Fourier modes. The forcing introduced in the largest wavenumbers $k < 1.5$ was computed so as to inject a constant level of energy in time. The statistics were obtained after a period of stabilization. The Taylor Reynolds number is approximately constant and equal to $R_\lambda = 200$.

Several other statistics have been compared for the same simulations. In general, our model seems to give better results than the two other LES models. The fig. 3 shows the comparison of the Probability Density Function of velocity increments. Looking for the results, our model seems to better capture the right shape of the PDF which is linked to the intermittency level.

An other statistics can also be performed on the velocity gradient tensor $\overline{A} = \partial \overline{u_i}/\partial x_j$. We compared the joint PDF of the two normalized invariant:

$$Q^\star = -\frac{1}{2} \frac{\overline{A}_{im}\overline{A}_{mi}}{\langle \overline{S}_{ij}\overline{s}_{ij}\rangle} \quad (9)$$

$$R^\star = -\frac{1}{3} \frac{\overline{A}_{im}\overline{A}_{mk}\overline{A}_{ki}}{\langle \overline{s}_{ij}\overline{s}_{ij}\rangle^{3/2}} \quad (10)$$

where $\overline{S}_{ij} = \frac{1}{2}(\overline{A}_{ij} + \overline{A}_{ji})$ and $\overline{s}_{ij} = \overline{S}_{ij} - \langle \overline{S}_{ij}\rangle$ ($\langle\cdot\rangle$ is a sample averaging). These statistics have already been used to discriminate between different turbulent models (van der Bos

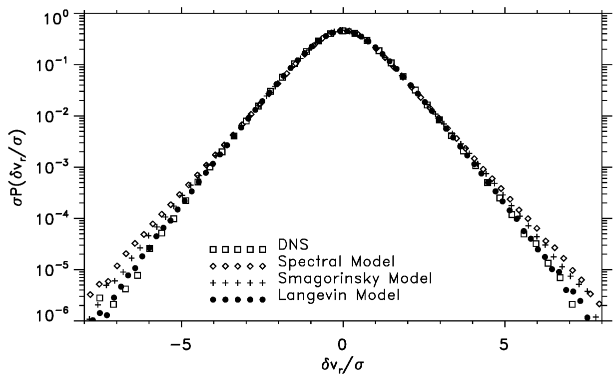

Figure 3: Comparison of the normalized PDF of the transversal velocity increments $(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x} + \delta\mathbf{x})$ with $\mathbf{u} \cdot \delta\mathbf{x} = 0$ and $|\delta\mathbf{x}| = L/64)$. The simulations are identical to fig. 2

et al., 2002). The results are shown fig. 4. The joint PDF is plotted for the DNS and the deviation from this PDF is plotted for the three LES simulations. The statistics with our model is, again, in better agreement with the DNS.

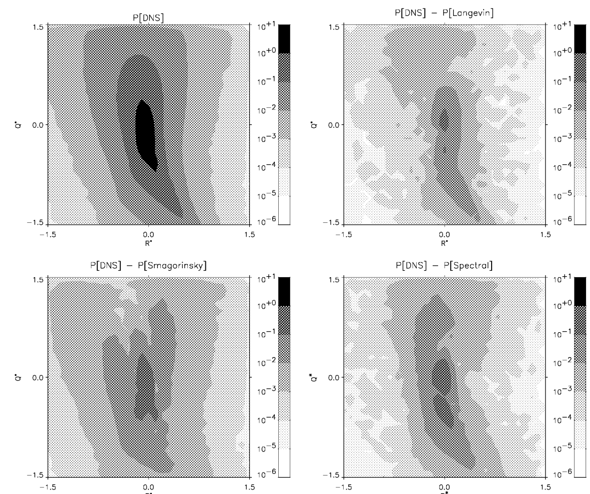

Figure 4: Joint PDF of $R^\star$ and $Q^\star$ (eqs. 9-10) for the DNS (upper left). The three other graphs present the difference of the same statistic between the DNS and the three other models (see description in fig. 2): Langevin model (upper right), Smagorinsky (lower left) and Spectral model (lower right).

CONCLUSION

In this work we presents a new strategy to derive a LES like model. This models falls in the category of models with stochastic evaluation of subfilter scales. Several models of this category have already been proposed (see (Domaradzki and Adams, 2002) for a review). In some case, the small scale velocity is estimated using the small scales generated by nonlinear interactions of resolved scales rescaled using a characteristic time (Adams and Stolz, 2001). In our approach, we chose to integrate an equation directly for $\boldsymbol{\ell}$ and using an additional model (turbulent viscosity in this case) for the remaining Reynolds stress tensor. The equation of $\boldsymbol{\ell}$ is derived from the Navier Stokes equation for the subgrid scales velocity using the RDT hypothesis of predominance of non-local interactions between resolved and subgrid scales over local interactions between subgrid scales. The derivation leads to a Langevin equation for $\boldsymbol{\ell}$. In a simplified version of the model, this equation can be replaced by new Langevin equation with a characteristic time τ which needs to be evaluated with respect to resolved scales quantities. The two versions of the model have been compared with equivalent high resolution DNS and more common LES models in the case of decaying and forced homogeneous isotropic turbulence. The model seems to reproduce accurately the main statistics of the flow. This approach can be easily extend to more realistic flows such as shear flows or flows with rotation. The main hypothesis of the models have already been studied theoretically for plane parallel flows for instance. The adaptation of the model to new flow configurations is in progress.

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