DIFFERENT EXPLICIT ALGEBRAIC REYNOLDS STRESS MODEL REPRESENTATIONS AND THEIR PREDICTIONS OF FULLY DEVELOPED TURBULENT ROTATING PIPE FLOW

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ABSTRACT

The least square projection method is used for obtaining optimal EARSMs for different incomplete sets of basis tensors. The possible singular behaviour depending on the choice of the basis tensors has been investigated. It is demonstrated that many of the incomplete representations, expressed in general 3D mean flows, have singularities in some specific flows, such as general 2D mean flows or more specifically, strain and/or rotation free 2D mean flows.

The different representations are investigated by computing fully developed rotating pipe flow. It is demonstrated that EARSMs containing basis tensors of odd powers higher than one, of the mean velocity gradients, do indeed predict a parabolic mean azimuthal velocity profile. The predcitions made by the incomplete representations deviate significantly from those by the complete representations, and the actual choice of basis tensors as well as the model parameters have a significant effect on the predictions.

INTRODUCTION

An explicit algebraic Reynolds stress model (EARSM) provides an interesting level of modelling by synthesizing the advantages of two-equation eddy-viscosity based models and full differential Reynolds stress models (DRSMs). While an EARSM has a constitutive relation that is explicitly dependent on the mean velocity gradients as an eddy viscosity based model and therefore not being too computationally cumbersome, the actual formulation is based on the full DRSM equations and hence, more of the flow physics can be incorporated. The EARSM itself is achieved by representing the solution of the implicit algebraic Reynolds stress model relation (ARSM), see Rodi (1976), in terms of a set of basis tensors. For a general flow five such independent basis tensors are needed to achieve a complete representation. In the present work we have used the least square based projection method, see for instance Jongen and Gatski (1998), which makes it possible to use fewer than five tensors. This representation is then optimal in a least square sense. Several representations based on different sets of basis tensors are discussed below.

In order to test the performance of the different representations, computations of fully developed turbulent rotating pipe flow have been performed. This particular case has been chosen on the basis of constituting a flow with a strong 3D character which is dependent on one spatial coordinate only and hence being relatively easy to implement. Rotating pipe flow has been studied by for instance Pettersson et al. (1998), Jakirlić et al. (2002), Grundestam et al. (2005) and Imao et al. (1996). This flow case has two characteristic features. The first is the increasingly more laminar-like axial velocity profile due to the increased axial rotation of the pipe. The second is the parabolic profile of the azimuthal velocity. Since this test case is strongly three-dimensional, the particular choice of basis tensors has a significant effect on the predictions of these two features, especially the parabolic shape of the azimuthal velocity profile, why this test case should be of particular interest in a study like this.

Governing equations

Fully developed rotating pipe flow is a 3D mean flow which is dependent on one spatial coordintate only, r, in a cylindrical coordinate system (r, θ, z). The mean velocity flow field is constrained to the axial and azimuthal directions. The Reynolds averaged Navier Stokes equations for the azimuthal and axial mean velocities are given by

$$\frac{\partial U_\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\nu\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) - r^2\overline{u_r u_\theta}\right] \quad (1)$$

$$\frac{\partial U_z}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[\nu r\frac{\partial U_z}{\partial r} - r\overline{u_r u_z}\right] \quad (2)$$

In the present work the Reynolds stresses, $\overline{u_i u_j}$, are evaluated using an explicit relation (EARSM) between the normalized mean strain and rotation rate tensors

$$S_{ij} = \frac{\tau}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \quad \Omega_{ij} = \frac{\tau}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \quad (3)$$

and the Reynolds stress anisotropy, $a_{ij} = \overline{u_i u_j}/K - 2\delta_{ij}/3$ where $\tau = K/\varepsilon$ is the turbulence time-scale.

An EARSM is derived from the transport equation for the Reynolds stress anisotropy which in a nonrotating inertial frame of reference reads

$$K\frac{Da_{ij}}{Dt} - \underbrace{\left(\frac{\partial T_{ijl}}{\partial x_l} - \frac{\overline{u_i u_j}}{K}\frac{\partial T_l^{(K)}}{\partial x_l}\right)}_{KD_{ij}^{(a)}} =$$

$$-\frac{\overline{u_i u_j}}{K}(\mathcal{P} - \varepsilon) + \mathcal{P}_{ij} - \varepsilon_{ij} + \Pi_{ij} \quad (4)$$

in cartesian tensor notation. D/Dt is the advective derivative defined as $D/Dt = \partial/\partial t + U_j \partial/\partial x_j$ in cartesian coordinates.

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$T_{ijl} = \overline{u_i u_j u_l} + (\overline{pu_i}\delta_{jl} + \overline{pu_j}\delta_{il})/\rho$ represents the total diffusion of the Reynolds stress components. $T_l^{(K)}$ is the diffusion of the turbulence kinetic energy, $K = \overline{u_i u_i}/2$, and is given by $T_l^{(K)} = T_{iil}/2$. The dissipation rate tensor, ε_{ij}, and the pressure strain rate tensor, Π_{ij}, need to be modelled while the Reynolds stress production $\mathcal{P}_{ij}$ and the turbulence kinetic energy production $\mathcal{P}$ can be expressed explicitly in a_{ij}, K, S_{ij} and Ω_{ij}. The pressure strain rate and dissipation rate anisotropy, $e_{ij} = \varepsilon_{ij}/\varepsilon - 2\delta_{ij}/3$ $(\varepsilon = \varepsilon_{ii}/2)$, tensors can be lumped together and the most general quasilinear model reads

$$\frac{\mathbf{\Pi}}{\varepsilon} - \mathbf{e} = -\frac{1}{2}\left(C_1^0 + C_1^1\frac{\mathcal{P}}{\varepsilon}\right)\mathbf{a} + C_2\mathbf{S}$$
$$+\frac{C_3}{2}(\mathbf{aS} + \mathbf{Sa} - \frac{2}{3}\{\mathbf{aS}\}\mathbf{I}) - \frac{C_4}{2}(\mathbf{a\Omega} - \mathbf{\Omega a}) \quad (5)$$

in boldface matrix notation in which $\mathcal{P}/\varepsilon \equiv -\{\mathbf{aS}\}$, where $\{\}$ denotes the trace and $\mathbf{I}$ is the identity matrix. The production of the Reynolds stresses normalized with the dissipation rate, can be expressed as

$$\frac{\boldsymbol{\mathcal{P}}}{\varepsilon} = -\frac{4}{3}\mathbf{S} - (\mathbf{aS} + \mathbf{Sa}) + \mathbf{a\Omega} - \mathbf{\Omega a} \quad (6)$$

The modelled transport equation of the Reynolds stress anisotropy can now be written

$$\tau\left(\frac{D\mathbf{a}}{Dt} - \boldsymbol{\mathcal{D}}^{(a)}\right) = A_0\Big((A_3 + A_4\frac{\mathcal{P}}{\varepsilon})\mathbf{a} + A_1\mathbf{S} -$$
$$(\mathbf{a\Omega} - \mathbf{\Omega a}) + A_2(\mathbf{aS} + \mathbf{Sa} - \frac{2}{3}\{\mathbf{aS}\}\mathbf{I})\Big) \quad (7)$$

The relations between the A and C-coefficients are

$$A_0 = \tfrac{C_4}{2} - 1 \quad A_1 = \tfrac{3C_2 - 4}{3A_0} \quad A_2 = \tfrac{C_3 - 2}{2A_0}$$
$$A_3 = \tfrac{2 - C_1^0}{2A_0} \quad A_4 = \tfrac{-C_1^1 - 2}{2A_0} \quad (8)$$

An EARSM is based on the weak equilibrium assumption, Rodi (1976), which amounts to neglecting the advection and diffusion of a_{ij}, i.e. the left-hand side of (7). This yields a purely algebraic relation

$$N\mathbf{a} = -A_1\mathbf{S} + (\mathbf{a\Omega} - \mathbf{\Omega a}) - A_2(\mathbf{aS} + \mathbf{Sa} - \frac{2}{3}\{\mathbf{aS}\}\mathbf{I}) \quad (9)$$

where $N = A_3 + A_4\frac{\mathcal{P}}{\varepsilon}$.

The curvature correction proposed by Wallin and Johansson (2002) has been used in conjunction with all EARSM representations. This contribution is derived from the advection of the Reynolds stress anisotropy and can be written as

$$-\tau(\mathbf{a\Omega}^{(r)} - \mathbf{\Omega}^{(r)}\mathbf{a}) \quad (10)$$

where $\Omega^{(r)}$ represents the local rotation rate of the flow. Instead of neglecting the l.h.s. of (7) it is replaced by (10) which leads to an alternative algebraic approximation of the transport equation for a_{ij}. This yields a systematic improvement of the weak equilibrium assumption in that it gives an approximation of the neglection of the advection of a_{ij} in a local streamline-based system. For fully developed rotating pipe flow, the correction exactly represents the advection and can be written

$$\Omega_{ij}^{(r)} = -\epsilon_{ij3}\frac{U_\theta}{r} \quad (11)$$

With the present modelling, the correction (10) can easily be included through the transformation

$$\mathbf{\Omega} \rightarrow \mathbf{\Omega}^* = \mathbf{\Omega} - \tfrac{\tau}{A_0}\mathbf{\Omega}^{(r)} \quad (12)$$

In the following, * has been omitted for simplicity. To close the system of equations, the high-*Re* $K-\omega$ platform by Wilcox (1988) is used. The platform equations are described in appendix.

EARSM FORMULATION

To reach a complete EARSM formulation, (9) can be formally solved, see e.g. Gatski and Speziale (1993) and Wallin and Johansson (2000). The procedure used here is based on the least square method and is outlined below. This method has previously been used by for instance Jongen and Gatski (1998), to derive EARSMs. The least square method has the advantage of allowing representations that are not based on a complete set of basis tensors. Since the Reynolds stress anisotropy has five independent components for a full 3D mean flow, a complete representation needs five independet basis tensors. If fewer than five tensors are used, the representation will still be optimal in a least square sense in terms of the chosen tensor basis.

The idea is to expand the Reynolds stress anisotropy in terms of the chosen basis tensors as

$$\mathbf{a} = \sum_i^M \beta_i\mathbf{T}^{(i)} \quad (13)$$

The expansion, (13), is then used in the ARSM-equation, (9), to form the system of equations for the EARSM-expansion coefficients $\{\beta_i\}$. The system of equations is then achieved by forming the inner product of the corresponding ARSM-equation with each and everyone of the M basis tensors. By following for instance Jongen and Gatski (1998), it is realized that the system of equations is given by

$$\sum_{i=1}^M \alpha_i(-N(\mathbf{T}^{(i)}, \mathbf{T}^{(j)}) + 2(\mathbf{T}^{(i)}\mathbf{\Omega}, \mathbf{T}^{(j)})$$
$$-2A_2(\mathbf{T}^{(i)}\mathbf{S}, \mathbf{T}^{(j)})) = \alpha_0 A_1(\mathbf{S}, \mathbf{T}^{(j)}) \quad j = 1, \ldots, M \quad (14)$$

Where the inner product is given by

$$(\mathbf{A}, \mathbf{B}) = A_{kl}B_{kl} \quad (15)$$

which is equal to $\{\mathbf{AB}\}$ if either $\mathbf{A}$ or $\mathbf{B}$ is symmetric. (14) hence constitutes a system of M linear equations (in terms of N) for the M unknowns, $\{\beta_i = \alpha_i/\alpha_0\}$. By using Cramer's rule, $\alpha_i = det(\mathbf{D}^{(i)})$, can be obtained. $\mathbf{D}^{(i)}$ are $M \times M$ matrices given by

$$D_{kl}^{(i)} = \begin{cases} N\{\mathbf{T}^{(l)}\mathbf{T}^{(k)}\} - 2\{\mathbf{T}^{(l)}\mathbf{\Omega}, \mathbf{T}^{(k)}\} \\ +2A_2\{\mathbf{T}^{(l)}\mathbf{S}, \mathbf{T}^{(k)}\} & l \neq i \\ \\ -A_1\{\mathbf{S}, \mathbf{T}^{(k)}\} & l = i \end{cases} \quad (16)$$

As pointed out by Jongen and Gatski (1998), a necessary condition for a representation to exist is that the basis tensors are linearly independent. Otherwise the denominator of the expansion coefficients, α_0, becomes zero.

In addition to solving (14), a polynomial equation for N has to be solved. This equation will be of varying order depending on how many basis tensors that are used and whether the mean flow is 2D or 3D. In the present work, the 2D mean flow form of N is used as an approximation.

The basis tensors, motivation

The basis tensors used are conveniently based on the strain and rotation rate tensors, (17) and (18) respectively. For rotating pipe flow, the mean strain and rotation rate tensors evaluated in a non-rotating frame of reference, read

$$\mathbf{S} = \frac{\tau}{2}\begin{bmatrix} 0 & \frac{dU_\theta}{dr} - \frac{U_\theta}{r} & \frac{dU_z}{dr} \\ \frac{dU_\theta}{dr} - \frac{U_\theta}{r} & 0 & 0 \\ \frac{dU_z}{dr} & 0 & 0 \end{bmatrix} \quad (17)$$

$$\mathbf{\Omega} = \frac{\tau}{2}\begin{bmatrix} 0 & -\frac{dU_\theta}{dr} - \frac{U_\theta}{r} & -\frac{dU_z}{dr} \\ \frac{dU_\theta}{dr} + \frac{U_\theta}{r} & 0 & 0 \\ \frac{dU_z}{dr} & 0 & 0 \end{bmatrix} \quad (18)$$

Basis tensors often used for EARSM representation are

$$\begin{array}{ll} \mathbf{T}^{(1)} = \mathbf{S} & \mathbf{T}^{(2)} = \mathbf{S}^2 - \frac{1}{3}II_\mathbf{S}\mathbf{I} \\ \mathbf{T}^{(3)} = \mathbf{\Omega}^2 - \frac{1}{3}II_\mathbf{\Omega}\mathbf{I} & \mathbf{T}^{(4)} = \mathbf{S\Omega} - \mathbf{\Omega S} \\ \mathbf{T}^{(5)} = \mathbf{S}^2\mathbf{\Omega} - \mathbf{\Omega S}^2 & \mathbf{T}^{(6)} = \mathbf{S\Omega}^2 + \mathbf{\Omega}^2\mathbf{S} - \frac{2}{3}IV\mathbf{I} \\ \mathbf{T}^{(7)} = \mathbf{S}^2\mathbf{\Omega}^2 + \mathbf{\Omega}^2\mathbf{S}^2 - \frac{2}{3}V\mathbf{I} & \mathbf{T}^{(8)} = \mathbf{S\Omega S}^2 - \mathbf{S}^2\mathbf{\Omega S} \\ \mathbf{T}^{(9)} = \mathbf{\Omega S\Omega}^2 - \mathbf{\Omega}^2\mathbf{S\Omega} & \mathbf{T}^{(10)} = \mathbf{\Omega S}^2\mathbf{\Omega}^2 - \mathbf{\Omega}^2\mathbf{S}^2\mathbf{\Omega} \end{array} \quad (19)$$

with the invariants

$$II_\mathbf{S} = \{\mathbf{S}^2\} \quad II_\mathbf{\Omega} = \{\mathbf{\Omega}^2\} \quad III_\mathbf{S} = \{\mathbf{S}^3\}$$
$$IV = \{\mathbf{S\Omega}^2\} \quad V = \{\mathbf{S}^2\mathbf{\Omega}^2\} \quad (20)$$

As a_{ij} itself, the basis tensors (19) are symmetric and traceless. Severel authors have used the whole set or subsets of (19) to represent the solution of the ARSM equation, (9). Depending on the modelling assumptions, if for instance $A_2 = 0$ or not, 3D representations are conviniently expressed in a various number of basis tensors, see for instance Gatski and Speziale (1993), Taulbee (1992) and Wallin and Johansson (2000). It is however possible to choose a set of five basis tensors and express any extra basis tensors in terms of these five, see Taulbee et al. (1994).

As mentioned above, one of the most characteristic features of turbulent rotating pipe flow is the parabolic shape of the azimuthal velocity, U_θ, profile. To be able to predict a U_θ-profile that deviates from solid body rotation, the consitutive relation must dictate a nonzero $a_{r\theta}$ for solid body rotation. This was realized by for instance Hirai et al. (1988), Wallin and Johansson (2000) etc. The basis tensors having a nonzero $r\theta$-components are $\mathbf{T}^{(1)}$, $\mathbf{T}^{(5)}$, $\mathbf{T}^{(6)}$ and $\mathbf{T}^{(10)}$. In fact, it can be readily shown that basis tensors of odd powers have a nonzero $r\theta$-component. However, by inspecting (17) one can draw the conclusion that for solid body rotation, $S_{r\theta} = 0$, and hence an EARSM using only $\mathbf{T}^{(1)}$ of these four basis tensors will indeed predict solid body rotation. A tensor containing $\mathbf{S}^3$ can be reduced using the Caley-Hamilton theorem to $II_\mathbf{S}\mathbf{S}/2$ (since $III_\mathbf{S} = 0$ for this particular case) and therefore do not itself give a deviation from solid body rotation. Hence, to be able to achieve a parabolic U_θ-profile, at least one of the basis tensors $\mathbf{T}^{(5)}$, $\mathbf{T}^{(6)}$ and $\mathbf{T}^{(10)}$ has to be used in the EARSM representation. Thus it is not necessary to include cubic tensor bases in order to capture the parabolic azimuthal velocity profile as long as $\mathbf{T}^{(10)}$ is used.

In order to solve the ARSM equation exactly for 2D mean flows, two or three basis tensors are needed depending on whether $A_2 = 0$ or not. If $A_2 = 0$ using the basis $\{\mathbf{T}^{(1)}, \mathbf{T}^{(4)}\}$ solves (9) exactly. If $A_2 \neq 0$, either $\mathbf{T}^{(2)}$ or $\mathbf{T}^{(3)}$ must be included in order to achieve the exact solution of (9) for 2D mean flows. The least square projection method described above does, however, allow choosing a set of basis tensor that is not complete. With this in mind, we have chosen to include $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(4)}$ in all EARSM representations studied in the present work in order to have the same 2D mean flow formulation as Wallin and Johansson (2000) when $A_2 = 0$. In this way, use can be made of the consistency efforts in their work.

Something important to keep in mind is that while any five basis tensor set would give a complete representation of the solution to (9) and hence in some sense be exact, every such representation cannot give the same predictions. By using a tensor basis that does not include $\mathbf{T}^{(5)}$, $\mathbf{T}^{(6)}$ or $\mathbf{T}^{(10)}$, the corresponding EARSM will predict solid body rotation and not the parabolic profile predicted by an EARSM including any of these bases.

EARSM representations

Due to the amounts of algebra generated, the EARSM representations are not listed here. An EARSM representation is referred to by listing the number of the basis tensors within curly brackets. For instance, {146} means the representation based on $\{\mathbf{T}^{(1)}, \mathbf{T}^{(4)}, \mathbf{T}^{(6)}\}$. The EARSMs derived are divided into two different categories, those with $A_2 = 0$ and those with $A_2 \neq 0$. The latter have the subscript $_{A_2}$. Two sets of model parameters have been used, $[A_0 = -0.72, A_1 = 1.2, A_2 = 0, A_3 = 1.8, A_4 = 2.25]$ and $[A_0 = -0.8, A_1 = 1.22, A_2 = 0.47, A_3 = 0.88, A_4 = 2.37]$. The first parameter set have been adopted from Wallin and Johansson (2002). The second comes from the linearized version of the SSG-model (Speziale et al. (1991)) proposed by Gatski and Speziale (1993). N is evaluated by deriving and solving its governing equation for 2D mean flows for the respective EARSM-representation. In all cases this implies solving a third order polynomial equation. The following representations are prioritized in this paper: {14}, {124}, {134}, {145}, {146}, {1410} (based on $\mathbf{T}^{(1)}$, $\mathbf{T}^{(4)}$ and $\mathbf{T}^{(10)}$), {1469}, $\{145\}_{A_2}$ and $\{124\}_{A_2}$. Other representations that have been derived are $\{134\}_{A_2}$, $\{146\}_{A_2}$ and $\{1469\}_{A_2}$. The WJ-EARSM ({13469}), Wallin and Johansson (2002), is also discussed and used as a comparison. So is the EARSM based on all ten basis tensors, $\{all\}_{A_2}$, derived by Gatski and Speziale (1993). These representations do in fact correspond to the exact solution (in terms of N) of the ARSM-equation for a zero/nonzero A_2 respectively.

The derived representations are simplified as far as possible concerning common factors in the denominators and numerators. Still, although many of the derived 3D representations have singularities in the 2D mean flow limit or for certain types of 2D mean flows, they do have nonsingular 2D formulations that can be achieved by successive application of the 2D mean flow invariant properties, $IV = 0$, $III_\mathbf{S} = 0$ and $V = II_\mathbf{S}II_\mathbf{\Omega}/2$, in combination with cancelling common factors in the denominator and numerator of the 3D representations coefficients. From this perspective it is hence important to make a distinction between the 2D mean flow behaviour of a 3D representation and the reduced 2D form of that particular representation. Table 1 shows when different 3D representations are singular in some generic 2D mean flows. As indicated in Table 1 some of the represenations are singular for 2D mean flows that are strain or rotation free, $II_\mathbf{S} = 0$ and $II_\mathbf{\Omega} = 0$

	2D	$II_{\mathbf{S}} = 0$	$II_{\mathbf{\Omega}} = 0$
$\{14\}$		x	
$\{124\}$		x	x
$\{134\}$		x	x
$\{145\}$	x		
$\{146\}$	x		
$\{1410\}$	x		
$\{1469\}$	x		
$\{13469\}$			
$\{124\}_{A_2}$		x	x
$\{134\}_{A_2}$		x	x
$\{145\}_{A_2}$	x		
$\{146\}_{A_2}$	x		
$\{1469\}_{A_2}$	x		
$\{all\}_{A_2}$			

Table 1: Singular behaviour of different 3D representations

respectively. The case of having $II_{\mathbf{S}} = 0$ is relatively easy to handle since it follows from (9) that a_{ij} should go to zero. Having a singularity when $II_{\mathbf{\Omega}} = 0$, on the other hand, might be a bit more troublesome since this might cause division by zero without necessary implying a zero Reynolds stress anisotropy. Note that the exact solutions to (9), $\{13469\}$ and $\{all\}_{A_2}$, are nonsingular for the flow types discussed. It should be pointed out that these situations are just three of assumably many possible singular flow types.

The problem with singularities, for some representations in the 2D mean flow limit and for some others in a shear or rotation free 2D mean flow, seems to emanate from the fact that the basis tensors are no longer linearly independent for 2D mean flows. This can be avoided if the denomintor (α_0) of the 3D representation can be factorized and the vanishing factors can cancel with the same factor in the numerators (α_i). From this perspective the representation $\{13469\}$ works very well since the denominator can be shown to always be well behaved without any singularities. The representation $\{all\}_{A_2}$ is also free of singularities for the situations in table 1. However, the denominator of this representation is rather complicated and might be singular for other flows.

PREDICTIONS OF FULLY DEVELOPED ROTATING PIPE FLOW

Computations of fully developed rotating pipe flow have been performed for rotation numbers $Ro = U_\theta(R)/U_b = 0$, 0.5 and 1. $U_\theta(R)$ is the azimuthal velocity of the wall and U_b is the mean axial bulk velocity. The model predictions are compared to the experimental data by Imao et al. (1996). The flow predictions made by the different representations, vary significantly, see figures 1-3. Several of the representations make identical predictions, as explained in the caption of figure 1. Important to remember is that nonrotating pipe flow is a 2D mean flow why several of the 3D representations, see table 1, are singular and are hence not shown for $Ro = 0$. For the nonrotating case, the representations having a zero/nonzero A_2 respectively, are almost indistinguishable. Due to general singularity problems, some of the representations in table 1 are not shown.

The computed axial velocity profiles normalized with the mean bulk velocity, U_z/U_b, are shown in figure (1). From this point of view, $\{124\}_{A_2}$ and $\{all\}_{A_2}$ perform very well and the predictions are in good agreement with the experiment, while

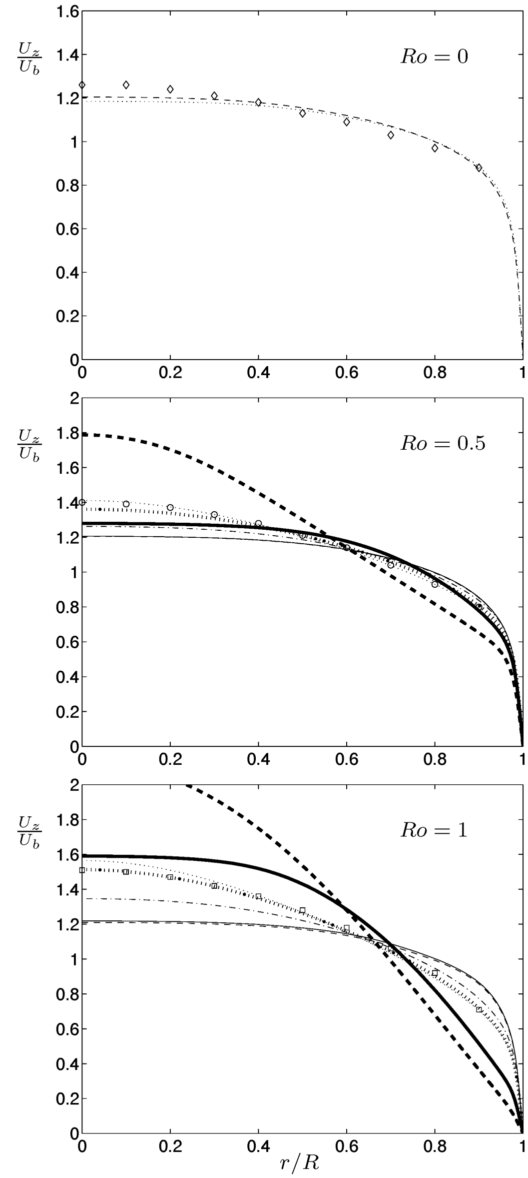

Figure 1: Normalized mean axial velocity, U_z/U_b. Representations: $\{124\}$, $\{134\}$ and $\{14\}$ ($--$) (thick); $\{145\}$ and $\{146\}$ ($-$); $\{1469\}$ and $\{13469\}$ ($-$) (thick); $\{1410\}$ ($--$); $\{124\}_{A_2}$ ($\cdots$); $\{145\}_{A_2}$ ($-\cdot$); $\{all\}_{A_2}$ ($\cdots$) (thick); experiments by Imao et al. ($\circ$).

$\{1469\}$ and $\{13469\}$ show a decent qualitative behaviour, although predicting a too flat profile in the center of the pipe for $Ro = 1$. The effect of rotation predicted by $\{124\}$ is far too drastic compared to the experiments. For $\{145\}$, $\{145\}_{A_2}$ and $\{1410\}$, on the other hand, the U_z-profiles are affected very little by the increased rotation.

The representations containing up to quadratic basis tensors are unable to predict the parabolic shape of the normalized U_θ-profile, see figure 2, as discussed above. Here, $\{1469\}$ and $\{145\}_{A_2}$, are in best agreement with the experiments. $\{145\}$ and $\{1410\}$ make similar predictions of the U_θ-profile and deviate too much from solid body rotation. $\{all\}_{A_2}$ gives a deviation from solid body rotation that is too small compared to the experiments. The predictions of $\{1410\}$ il-

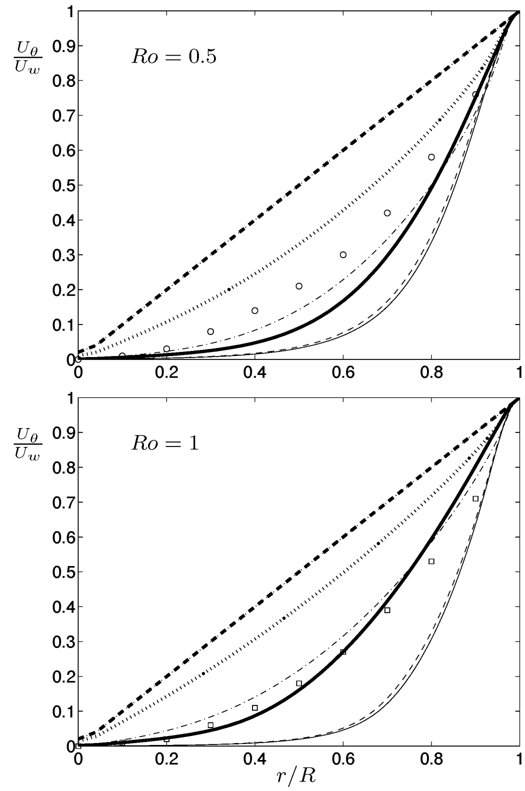

Figure 2: Azimutal mean velocity, U_θ/U_w. Legends as in figure 1.

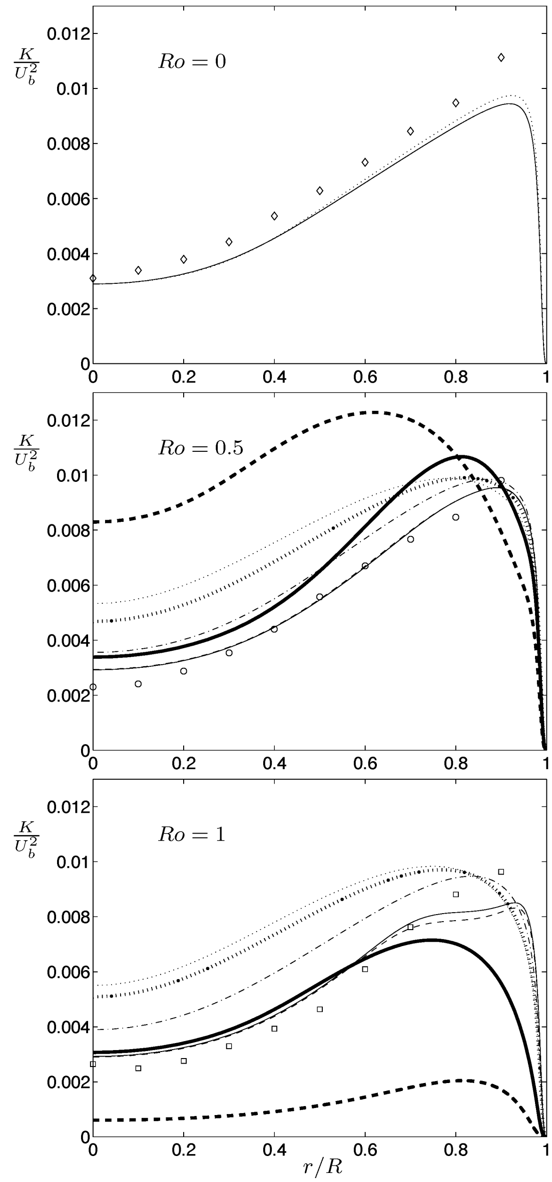

Figure 3: Turbulence kinetic energy, K/U_b^2. Legends as in figure 1.

lustrates that a parabolic mean azimuthal velocity profile can be obtained with an EARSM containing a fifth order instead of a cubic basis tensor.

The computed profiles of the turbulence kinetic energy normalized with the mean bulk axial velocity, K/U_b^2, are shown in figure 3. The representations $\{124\}$ and $\{124\}_{A_2}$ do not agree very well with the experiments in comparison to the other representations. $\{124\}$ shows a too dramatic decrease in K when the rotation is increased from $Ro = 0.5$ to $Ro = 1$. $\{all\}_{A_2}$ and $\{124\}_{A_2}$ has a tendency to overpredict the turbulence intensity in the center of the pipe for the rotating cases. This is also the case, although more modest, for $\{145\}_{A_2}$ and $Ro = 1$. The representations $\{145\}$ and $\{1410\}$ show a rather peculiar behaviour close to the wall for $Ro = 1$.

DISCUSSION / CONCLUSIONS

The least square projection method provides a straightforward way to derive an EARSM representation using an arbitrary set of basis tensors. However, singularity problems can arise. If the denominator of the representation coefficients cannot be factorized properly (i.e. having a common factor with the numerator) singular behaviour occurs when the characteristics of the flow make two or more of the basis tensors linearly dependent, see also Jongen and Gatski (1998). This is the case for many of the commonly used representations which, as demonstrated above, have a singular behaviour either for general 2D mean flows or strain or rotation free 2D flows. This is something that can cause problems in general CFD codes where the 3D mean flow representation is implemented and the flow at some location approaches the state where the constitutive relation becomes singular. However, it is not solely a matter of division by zero but rather having both a zero denominator and numerator. Therefore the singularity can in principle be addressed by considering the limiting value of the representation as the singularity is approached. This has not been considered in the present study. Nevertheless, these issues have to be considered when a new representation is implemented in a general CFD solver.

Another, perhaps more positive, aspect of the projection method is that the order of the polynimial equation for N can be reduced by using fewer basis tensors. Still, the equation for N is of third order for a representation with two basis tensors and increases with one order for every additional basis tensor that is included unless the expansion coefficients can be factorized. Factorization is unfortunately not possible for many of the representations.

For the rotating pipe flow, it is obvious that the choice of basis tensors has a significant effect on the predictions. For instance, by including either $\mathbf{T}^{(5)}$, $\mathbf{T}^{(6)}$ or $\mathbf{T}^{(10)}$ a velocity profile that deviates from solid body rotation is achieved. It is somewhat surprising that the biggest difference between $\{124\}_{A_2}$ and $\{all\}_{A_2}$ is the small deviation from solid body rotation that $\{all\}_{A_2}$ shows. It is also clear that the choice of model parameters play an important role. This is indicated when comparing $\{145\}$ with $\{145\}_{A_2}$ and $\{124\}$ with $\{124\}_{A_2}$, for instance. But also $\{all\}_{A_2}$ and $\{13469\}$ of course. Despite that some representations based on incomplete basis tensor sets, make better predictions than the complete representation from some perspectives (e.g. comparing $\{145\}_{A_2}$ and $\{all\}_{A_2}$ for U_θ), there is nothing that indicates that this should hold generally. The complete representations capture indeed the physics of the ARSM-equation most accurately while the diffences compared to the incomplete representations are purely an offspring of projecting the solution on an incomplete tensor basis. Therefore, the advantage of, for instance, $\{145\}_{A_2}$ over $\{all\}_{A_2}$ for U_θ is purely coincidential and any shortcomings of the complete representations should be addressed the underlying modelling or the weak equilibrium assumption.

Finally, one could conclude that although the least square projection method provides a convinient method of representing tensors in terms of chosen basis tensor, many of the EARSM representations obtained are too complicated and perhaps even ill behaved to be of practical use.

APPENDIX. THE $K-\omega$ - PLATFORM

The K and ω equations are given

$$\frac{DK}{Dt} = \mathcal{P} - \varepsilon + \mathcal{D}_K \tag{21}$$

$$\frac{\partial \omega}{\partial t} = \alpha \frac{\omega}{K}\mathcal{P} - \beta\omega^2 + \mathcal{D}_\omega \tag{22}$$

$\mathcal{P} = -\varepsilon\{\mathbf{aS}\}$ is the turbulence energy production and $\mathcal{D}$ represents the diffusion. Here, the diffusion model proposed by Daly and Harlow (1970) has been adopted. For the present flow case, this model takes the form

$$\mathcal{D}_K = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nu + c_s\frac{K}{\varepsilon}\overline{u_r u_r}\right)\frac{\partial K}{\partial r}\right] \tag{23}$$

$$\mathcal{D}_\omega = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nu + c_\omega\frac{K}{\varepsilon}\overline{u_r u_r}\right)\frac{\partial \omega}{\partial r}\right] \tag{24}$$

for K and ω repspectively. The model parameters used are $c_s = 0.11$ and $c_\omega = 0.11$, $\beta^* = 0.09$, $\alpha = 5/9$ and $\beta = 3/40$. The dissipation rate is given by $\varepsilon = \beta^* K\omega$.

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