MODELING OF THE TURBULENT MHD RESIDUAL-ENERGY EQUATION USING A STATISTICAL THEORY

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ABSTRACT

In the framework of magnetohydrodynamics (MHD), difference between the kinetic and magnetic energies is investigated. Deviation from the equipartition is measured by the turbulent MHD residual energy (K_R). With the aid of the TSDIA or a statistical analytical theory for inhomogeneous turbulence, expressions for the correlation tensors appearing in the K_R equation are derived. Using the results, we propose a model equation for K_R evolution. Through the examination of the K_R-equation structure, it is shown that the evolution of the scaled K_R is related to the cross helicity (velocity/magnetic-field correlation) of turbulence.

INTRODUCTION

Magnetohydrodynamics (MHD) provides a powerful tool for analyzing fluid-like behaviors of magnetized plasmas. In homogeneous isotropic MHD turbulence, an equipartition between the kinetic and magnetic fluctuation energies should be realized in the presence of a uniform magnetic field (Kraichnan, 1965). Conversely, it is not the case in inhomogeneous turbulence with a mean flow shear. The deviation from the equipartition plays an important role in various MHD shear flow phenomena.

In the turbulent dynamo or generation mechanism of large-scale magnetic fields in turbulence, one of the key mechanisms is called the helicity or α effect. It is known that the α effect does not solely depend on the kinetic helicity but on the residual helicity (difference between the kinetic and current helicities) (Pouquet et al., 1976). Since the magnitude of the residual helicity is linked with the counterpart of the residual energy (difference between the kinetic and magnetic energies), a proper estimate of the residual energy is indispensable for dynamo studies (Fig. 1).

Another noticeable example is the solar-wind turbulence, where deviation from equipartition between the kinetic and magnetic energies is ubiquitously observed. One approach to treat this deviation is to consider the evolution equations for both the Reynolds and the Maxwell stresses with all their intricacy. Another approach, after the eddy-viscosity-type modeling of MHD turbulence, is to define a statistical quantity called the turbulent MHD residual energy by $K_\mathrm{R} = \langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle /2$ ($\mathbf{u}'$: velocity fluctuation, $\mathbf{b}'$: magnetic-field fluctuation, $\langle \cdot \rangle$: ensemble average) and consider the evolution equation for K_R as well as the counterpart for the turbulent MHD energy $K (= \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle /2)$. In this work, we adopt the latter approach and discuss the deviation from equipartition from the viewpoint of the K_R-equation modeling.

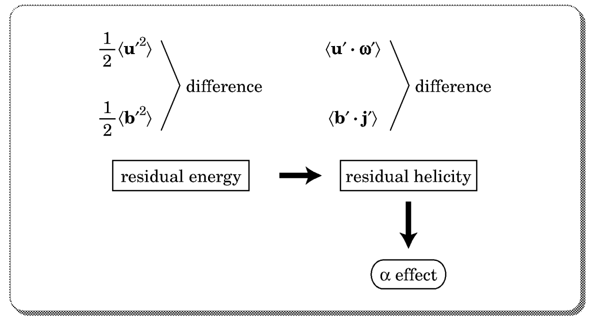

Figure 1: Residual energy and turbulent dynamo: A large difference between the kinetic and magnetic energies is related to a large amplitude of the residual helicity, leading to an effective α dynamo. $\omega (= \nabla \times \mathbf{u}')$: vorticity fluctuation, $\mathbf{j}' (= \nabla \times \mathbf{b}')$: current fluctuation.

STATISTICAL ANALYSIS AND ITS RESULTS

Fundamental equations

An incompressible magnetohydrodynamic plasma in a system rotating with the angular velocity $\boldsymbol{\Omega}_\mathrm{F}$ obeys

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - (\mathbf{b} \cdot \nabla)\, \mathbf{b} = -\nabla p_\mathrm{M} - 2\boldsymbol{\Omega}_\mathrm{F} \times \mathbf{u} + \nu \nabla^2 \mathbf{u} \quad (1)$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{b} - (\mathbf{b} \cdot \nabla)\, \mathbf{u} = \lambda \nabla^2 \mathbf{b} \quad (2)$$

with the solenoidal conditions for the velocity $\mathbf{u}$ and the magnetic field $\mathbf{b}$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0 \quad (3)$$

Here, p_M is the MHD pressure, ν is the kinematic viscosity, and λ is the magnetic diffusivity. We introduce the Elssässer variables:

$$\phi = \mathbf{u} + \mathbf{b}, \quad \psi = \mathbf{u} - \mathbf{b} \quad (4)$$

and rewrite Eqs. (1)-(3) with neglecting the difference between ν and λ, we have

$$\frac{\partial \phi}{\partial t} + (\psi \cdot \nabla)\, \phi = -\nabla p_\mathrm{M} - \boldsymbol{\Omega}_\mathrm{F} \times (\phi + \psi) + \frac{\nu + \lambda}{2} \nabla^2 \phi \quad (5)$$

$$\frac{\partial \psi}{\partial t} + (\phi \cdot \nabla)\, \psi = -\nabla p_\mathrm{M} - \boldsymbol{\Omega}_\mathrm{F} \times (\psi + \phi) + \frac{\nu + \lambda}{2} \nabla^2 \psi \quad (6)$$

$$\nabla \cdot \phi = \nabla \cdot \psi = 0 \quad (7)$$

Equations (5)-(7) show that the permutation of ϕ with ψ does not change the system of equations. We fully utilize this property in the following calculations.

Mean and fluctuation

We divide the field quantities f such as $\mathbf{u}$, $\mathbf{b}$, etc. into the mean F and the fluctuation f' around it as

$$f = F + f', \quad F = \langle f \rangle \tag{8}$$

where

$$f = (\mathbf{u}, \omega, \mathbf{b}, \mathbf{j}, p_{\mathrm{M}}, \phi, \psi) \tag{9}$$

$$F = (\mathbf{U}, \mathbf{\Omega}, \mathbf{B}, \mathbf{J}, P_{\mathrm{M}}, \mathbf{\Phi}, \mathbf{\Psi}) \tag{10}$$

$$f' = (\mathbf{u}', \omega', \mathbf{b}', \mathbf{j}', p'_{\mathrm{M}}, \phi', \psi') \tag{11}$$

where $\omega (= \nabla \times \mathbf{u})$ is the vorticity and $\mathbf{j} (= \nabla \times \mathbf{b})$ is the electric-current density. Substitution of Eq. (8) into Eqs. (1)-(3) or Eqs. (5)-(7) gives us the equations for the mean field and the counterparts for the fluctuations.

Equations for the turbulent residual energy

The turbulent MHD residual energy is defined by

$$K_{\mathrm{R}} \equiv \langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle / 2 \tag{12}$$

From Eqs. (1) and (2) and the Reynolds decomposition [Eq. (8)], the evolution equation of K_{R} is exactly given as

$$\begin{aligned}
\frac{DK_{\mathrm{R}}}{Dt} &\equiv \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) K_{\mathrm{R}} \\
&= -R_{\mathrm{T}}^{ab} \frac{\partial U^a}{\partial x^b} + W_{\mathrm{T}}^{ab} \frac{\partial B^a}{\partial x^b} - \mathbf{\Gamma} \cdot \mathbf{B} \\
&\quad -\nu \left\langle \frac{\partial u'^a}{\partial x^b} \frac{\partial u'^a}{\partial x^b} \right\rangle + \lambda \left\langle \frac{\partial b'^a}{\partial x^b} \frac{\partial b'^a}{\partial x^b} \right\rangle - \frac{\partial}{\partial x^a} \left\langle u'^a p'_{\mathrm{M}} \right\rangle \\
&\quad -\frac{\partial}{\partial x^b} \left\langle u'^b \frac{1}{2} \left(u'^{a\,2} - b'^{a\,2} \right) \right\rangle \\
&\quad +\frac{\partial^2}{\partial x^b \partial x^b} \left(\nu \frac{1}{2} \left\langle u'^{a\,2} \right\rangle - \lambda \frac{1}{2} \left\langle b'^{a\,2} \right\rangle \right) \\
&\quad + \left\langle b'^b \left(u'^a \frac{\partial b'^a}{\partial x^b} - b'^a \frac{\partial u'^a}{\partial x^b} \right) \right\rangle
\end{aligned} \tag{13}$$

where $\mathbf{U}$ is the mean velocity and $\mathbf{B}$ is the mean magnetic field. Here, $\mathbf{R}_{\mathrm{T}}$ and $\mathbf{W}_{\mathrm{T}}$ denote the total MHD self- and cross-correlation tensors, respectively, and $\mathbf{\Gamma}$ is the torsional cross vector. They are defined as

$$R_{\mathrm{T}}^{\alpha\beta} = \left\langle u'^\alpha u'^\beta + b'^\alpha b'^\beta \right\rangle \tag{14}$$

$$W_{\mathrm{T}}^{\alpha\beta} = \left\langle u'^\alpha b'^\beta + u'^\beta b'^\alpha \right\rangle \tag{15}$$

$$\Gamma^\alpha = \left\langle b'^a \frac{\partial u'^a}{\partial x^\alpha} + u'^a \frac{\partial b'^a}{\partial x^\alpha} \right\rangle \tag{16}$$

Statistical Analysis

With the aid of the two-scale direct-interaction approximation (TSDIA) (Yoshizawa, 1998), a statistical analytical theory for inhomogeneous turbulence, we explore the expressions for these correlation functions [Eqs. (14)-(16)]. The formal procedure is summarized as follows.

Introduction of two scales. Using a scale parameter δ, we introduce the slow and rapid variables:

$$\xi = \mathbf{x}, \;\; \mathbf{X} = \delta \mathbf{x}; \;\; \tau = t, \;\; T = \delta t \tag{17}$$

The slow variable $(\mathbf{X}, T)$ provide long spatial and temporal scales since their changes are not negligible only when $\mathbf{x}$ and t are large. On the other hand, the rapid variables (ξ, τ) is appropriate for describing the fine spatio-temporal motions. With these two-scale variables, the spatial and temporal derivatives are expressed as

$$\nabla = \nabla_\xi + \delta \nabla_{\mathbf{X}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta \frac{\partial}{\partial T} \tag{18}$$

and the field quantities f are divided into F and f' as

$$f = F(\mathbf{X}; T) + f'(\xi, \mathbf{X}; \tau, T) \tag{19}$$

Fourier representations. We perform the Fourier transform with respect to the rapid variable ξ and express the governing equations in the wave space. For instance, the equation for ϕ' is expressed as

$$\begin{aligned}
&\frac{\partial \phi'^\alpha(\mathbf{k}; \tau)}{\partial t} + \nu k^2 \phi'^\alpha(\mathbf{k}; \tau) - i k^\alpha p'_{\mathrm{M}}(\mathbf{k}; \tau) \\
&-ik^b \iint \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) d\mathbf{p} d\mathbf{q} \psi'^b(\mathbf{p}; \tau) \phi'^\alpha(\mathbf{q}; \tau) \\
&= i(\mathbf{k} \cdot \mathbf{B}) \phi'^\alpha(\mathbf{k}; \tau) - \varepsilon^{\alpha a b} \Omega_{\mathrm{F}}^a \left(\phi'^b(\mathbf{k}; \tau) + \psi'^b(\mathbf{k}; \tau) \right) \\
&+\delta \left(-\psi'^a(\mathbf{k}; \tau) \frac{\partial \Phi^\alpha}{\partial X^a} - \frac{D\phi'(\mathbf{k}; \tau)}{DT_{\mathrm{I}}} \right. \\
&+B^a \frac{\partial \phi'^\alpha(\mathbf{k}; \tau)}{\partial X_{\mathrm{I}}^a} - \frac{\partial p'_{\mathrm{M}}(\mathbf{k}; \tau)}{\partial X_{\mathrm{I}}^\alpha} \\
&\left. -\frac{\partial}{\partial X_{\mathrm{I}}^a} \iint \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) d\mathbf{p} d \psi'^a(p; \tau) \phi'^\alpha(q; \tau) \right)
\end{aligned} \tag{20}$$

$$\mathbf{k} \cdot \phi'_{\mathrm{S}}(\mathbf{k}; \tau) = 0 \tag{21}$$

where

$$\phi'(\mathbf{k}; \tau) = \phi'_{\mathrm{S}}(\mathbf{k}; \tau) + \delta \left(-i \frac{\mathbf{k}}{k^2} \frac{\partial \phi'^a(\mathbf{k}; t)}{\partial X_{\mathrm{I}}^a} \right) \tag{22}$$

and $\delta(\mathbf{k} - \mathbf{q} - \mathbf{r})$ in Eq. (20) denotes the delta function which vanishes unless the wave-vector relation $\mathbf{k} = \mathbf{q} + \mathbf{r}$ is satisfied.

Scale-parameter expansion. We expand the field quantities $\vartheta' = (\phi', \psi')$ in the scale parameter δ:

$$\vartheta' = \vartheta'_0 + \delta \vartheta'_1 + \delta^2 \vartheta'_2 + \cdots \tag{23}$$

where ϑ_0 is the field without the mean field. We further expand this in the external-field parameter such as the mean magnetic field $\mathbf{B}$ and the angular velocity $\mathbf{\Omega}_{\mathrm{F}}$:

$$\vartheta' = \vartheta'_{\mathrm{B}} + \vartheta'_{01} + \vartheta'_{02} + \cdots + \vartheta'_1 + \vartheta'_2 + \cdots \tag{24}$$

Here, ϑ'_{B} is the basic field corresponding to the homogeneous isotropic turbulence. For instance, the equation for ϕ'_{B} is written as

$$\begin{aligned}
&\frac{\partial {\phi'_{\mathrm{B}}}^\alpha(\mathbf{k}; \tau)}{\partial \tau} + \nu k^2 {\phi'_{\mathrm{B}}}^\alpha - i k^a D^{\alpha b}(\mathbf{k}) \times \\
&\iint \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) d\mathbf{p} d\mathbf{q} {\psi'_{\mathrm{B}}}^a(\mathbf{p}; \tau) {\phi'_{\mathrm{B}}}^b(\mathbf{q}; \tau) = 0
\end{aligned} \tag{25}$$

where $D^{\alpha\beta}(=\delta^{\alpha\beta}-k^{\alpha}k^{\beta}/k^2)$ is the projection operator in the wave space. Note that this equation is the same as the counterpart for the homogeneous isotropic turbulence except the implicit dependence on the slow variable $\mathbf{X}$ and T.

Calculation using the Green's functions. We define the Green's functions for $\phi'_{\rm B}$, ϕ'_{01}, etc. For example, the one for $\phi_{\rm B}$, ${G'_\phi}^{\alpha\beta}(\mathbf{k};\tau,\tau')$, is defined by

$$\frac{\partial {G'_\phi}^{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial\tau}+\nu k^2{G'_\phi}^{\alpha\beta}(\mathbf{k};\tau,\tau')-ik^aD^{\alpha b}(\mathbf{k})\times \iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}{\psi'_{\rm B}}^{a}(\mathbf{p};\tau){G'_\phi}^{b\beta}(\mathbf{q};\tau,\tau')=0 \quad (26)$$

Using these Green's functions, we formally solve ϑ'_{01} and ϑ'_1.

Statistical properties for the basic fields. Since the basic fields are homogeneous and isotropic, we assume the statistical properties for them in the form:

$$\frac{\left\langle\vartheta^\alpha_{\rm B}(\mathbf{k};\tau)\chi^\beta_{\rm B}(\mathbf{k}';\tau')\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}=D^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi}(\mathbf{k};\tau,\tau')+\frac{i}{2}\frac{k^a}{k^2}\epsilon^{\alpha\beta a}H_{\vartheta\chi}(\mathbf{k};\tau,\tau') \quad (27)$$

$$\left\langle{G'_\vartheta}^{\alpha\beta}(\mathbf{k};\tau,\tau')\right\rangle=\delta^{\alpha\beta}G_\vartheta(\mathbf{k};\tau,\tau') \quad (28)$$

where ϑ and χ denote ϕ and/or ψ. For later convenience, we introduce the symmetric and antisymmetric parts of the Green's functions as

$$G_{\rm S}(\mathbf{k};\tau,\tau')=\frac{1}{2}\left(G_\phi(\mathbf{k};\tau,\tau')+G_\psi(\mathbf{k};\tau,\tau')\right) \quad (29)$$

$$G_{\rm A}(\mathbf{k};\tau,\tau')=\frac{1}{2}\left(G_\phi(\mathbf{k};\tau,\tau')-G_\psi(\mathbf{k};\tau,\tau')\right) \quad (30)$$

Calculation of the correlation functions. Following the above procedure, we calculate the correlation functions Eqs. (14)-(16) with

$$\left\langle\phi'^\alpha\phi'^\beta\right\rangle=\left\langle{\phi'_{\rm B}}^\alpha{\phi'_{\rm B}}^\beta\right\rangle+\left\langle{\phi'_{\rm B}}^\alpha{\phi'_{01}}^\beta\right\rangle+\left\langle{\phi'_{01}}^\alpha{\phi'_{\rm B}}^\beta\right\rangle+\cdots+\left\langle{\phi'_{\rm B}}^\alpha{\phi'_1}^\beta\right\rangle+\left\langle{\phi'_1}^\alpha{\phi'_{\rm B}}^\beta\right\rangle+\cdots \quad (31)$$

$$\left\langle\psi'^\alpha\psi'^\beta\right\rangle=\left\langle{\psi'_{\rm B}}^\alpha{\psi'_{\rm B}}^\beta\right\rangle+\left\langle{\psi'_{\rm B}}^\alpha{\psi'_{01}}^\beta\right\rangle+\left\langle{\psi'_{01}}^\alpha{\psi'_{\rm B}}^\beta\right\rangle+\cdots+\left\langle{\psi'_{\rm B}}^\alpha{\psi'_1}^\beta\right\rangle+\left\langle{\psi'_1}^\alpha{\psi'_{\rm B}}^\beta\right\rangle+\cdots \quad (32)$$

$$\left\langle\phi'^\alpha\frac{\partial\psi'^\beta}{\partial x^b}\right\rangle=\left\langle{\phi'_{\rm B}}^\alpha\frac{\partial{\psi'_{\rm B}}^\beta}{\partial x^b}\right\rangle+\left\langle{\phi'_{\rm B}}^\alpha\frac{\partial{\psi'_{01}}^\beta}{\partial x^b}\right\rangle+\left\langle{\phi'_{01}}^\alpha\frac{\partial{\psi'_{\rm B}}^\beta}{\partial x^b}\right\rangle+\cdots+\left\langle{\phi'_{\rm B}}^\alpha\frac{\partial{\psi'_1}^\beta}{\partial x^b}\right\rangle+\left\langle{\phi'_1}^\alpha\frac{\partial{\psi'_{\rm B}}^\beta}{\partial x^b}\right\rangle+\cdots \quad (33)$$

$$\left\langle\psi'^\alpha\frac{\partial\phi'^\beta}{\partial x^b}\right\rangle=\left\langle{\psi'_{\rm B}}^\alpha\frac{\partial{\phi'_{\rm B}}^\beta}{\partial x^b}\right\rangle+\left\langle{\psi'_{\rm B}}^\alpha\frac{\partial{\phi'_{01}}^\beta}{\partial x^b}\right\rangle+\left\langle{\psi'_{01}}^\alpha\frac{\partial{\phi'_{\rm B}}^\beta}{\partial x^b}\right\rangle+\cdots+\left\langle{\psi'_{\rm B}}^\alpha\frac{\partial\phi'1^\beta}{\partial x^b}\right\rangle+\left\langle{\psi'_1}^\alpha\frac{\partial{\phi'_{\rm B}}^\beta}{\partial x^b}\right\rangle+\cdots \quad (34)$$

Results from TSDIA

With the abbreviation notations:

$$I_n\left\{A\right\}=\int dkk^{2n}A(k;\tau,\tau) \quad (35)$$

$$I_n\left\{A,B\right\}=\int dkk^{2n}\int_{-\infty}^{\tau}d\tau_1A(k;\tau,\tau_1)B(k;\tau,\tau_1) \quad (36)$$

the results of the TSDIA analysis are expressed as

$$\begin{aligned}R^{\alpha\beta}_{\rm T}&=\left\langle u'^\alpha u'^\beta+b'^\alpha b'^\beta\right\rangle=\frac{1}{2}\left\langle\phi'^\alpha\phi'^\beta+\psi'^\alpha\psi'^\beta\right\rangle\\&=\frac{2}{3}\delta^{\alpha\beta}I_0\left\{Q_{uu}+Q_{bb}\right\}\\&\quad-\frac{7}{15}I_0\left\{G_{\rm S},(Q_{uu}-Q_{bb})\right\}S^{\alpha\beta}\\&\quad-\frac{7}{15}I_0\left\{G_{\rm S},(Q_{bu}-Q_{ub})\right\}M^{\alpha\beta}\\&\quad+2\Omega^\alpha_{\rm F}\Lambda^\beta_R+2\Omega^\beta_{\rm F}\Lambda^\alpha_R+6\delta^{\alpha\beta}2\mathbf{\Omega}_{\rm F}\cdot\mathbf{\Lambda}_R\end{aligned} \quad (37)$$

$$\begin{aligned}W^{\alpha\beta}_{\rm T}&=\left\langle u'^\alpha b'^\beta+b'^\alpha u'^\beta\right\rangle=\frac{1}{2}\left\langle\phi'^\alpha\phi'^\beta-\psi'^\alpha\psi'^\beta\right\rangle\\&=\frac{2}{3}\delta^{\alpha\beta}I_0\left\{Q_{ub}+Q_{bu}\right\}\\&\quad-\frac{7}{15}I_0\left\{G_{\rm S},(Q_{bu}-Q_{ub})\right\}S^{\alpha\beta}\\&\quad-\frac{7}{15}I_0\left\{G_{\rm S},(Q_{uu}-Q_{bb})\right\}M^{\alpha\beta}\\&\quad+2\Omega^\alpha_{\rm F}\Lambda^\beta_W+2\Omega^\beta_{\rm F}\Lambda^\alpha_W+6\delta^{\alpha\beta}2\mathbf{\Omega}_{\rm F}\cdot\mathbf{\Lambda}_W\end{aligned} \quad (38)$$

$$\begin{aligned}\Gamma^\alpha&=\left\langle b'^a\frac{\partial u'^a}{\partial x^\alpha}-u'^a\frac{\partial b'^a}{\partial x^\alpha}\right\rangle=\frac{1}{2}\left\langle\phi'^a\frac{\partial\psi'^a}{\partial x^\alpha}-\psi'^a\frac{\partial\phi'^a}{\partial x^\alpha}\right\rangle\\&=\frac{4}{3}I_1\left\{G_{\rm S},(Q_{uu}-Q_{bb})\right\}B^\alpha\\&\quad+\frac{1}{3}I_0\left\{G_{\rm S},(H_{uu}+H_{bb})\right\}(\nabla\times\mathbf{B})^\alpha\\&\quad-\frac{1}{3}I_0\left\{G_{\rm S},(H_{ub}+H_{bu})\right\}(\mathbf{\Omega}+2\mathbf{\Omega}_{\rm F})^\alpha\end{aligned} \quad (39)$$

where

$$\mathbf{\Lambda}_R=\frac{1}{15}\left(I_{-1}\left\{G_{\rm S},\nabla H_{uu}\right\}+I_{-1}\left\{G_{\rm A},\nabla H_{ub}\right\}\right) \quad (40)$$

$$\mathbf{\Lambda}_W=\frac{1}{15}\left(I_{-1}\left\{G_{\rm S},\nabla H_{ub}\right\}+I_{-1}\left\{G_{\rm A},\nabla H_{uu}\right\}\right) \quad (41)$$

Here, $\mathbf{S}$ and $\mathbf{M}$ are the velocity and magnetic strain rates defined by

$$S^{\alpha\beta}\equiv\frac{\partial U^\beta}{\partial x^\alpha}+\frac{\partial U^\alpha}{\partial x^\beta},\quad M^{\alpha\beta}\equiv\frac{\partial B^\beta}{\partial x^\alpha}+\frac{\partial B^\alpha}{\partial x^\beta} \quad (42)$$

Note that, in Eqs. (37)-(41), Q_{uu}, H_{uu}, etc. are the spectral functions that are related to the basic fields as

$$\begin{aligned}\left\langle\frac{1}{2}\left({\mathbf{u}'_{\rm B}}^2+{\mathbf{b}'_{\rm B}}^2\right)\right\rangle&=\int d\mathbf{k}\left(Q_{uu}(k;\tau,\tau)+Q_{bb}(k;\tau,\tau)\right)\\&=\frac{1}{2}\int d\mathbf{k}\left(Q_{\phi\phi}(k;\tau,\tau)+Q_{\psi\psi}(k;\tau,\tau)\right)\end{aligned} \quad (43)$$

$$\begin{aligned}\left\langle\mathbf{u}'_{\rm B}\cdot\mathbf{b}'_{\rm B}\right\rangle&=\int d\mathbf{k}Q_{ub}(k;\tau,\tau)\\&=\frac{1}{2}\int d\mathbf{k}\left(Q_{\phi\phi}(k;\tau,\tau)-Q_{\psi\psi}(k;\tau,\tau)\right)\end{aligned} \quad (44)$$

$$\left\langle -\mathbf{u}'_{\rm B} \cdot \omega'_{\rm B} + \mathbf{b}'_{\rm B} \cdot \mathbf{j}'_{\rm B} \right\rangle = \int d\mathbf{k} \left(-H_{uu}(k;\tau,\tau) + H_{bb}(k;\tau,\tau) \right)$$
$$= -\frac{1}{2} \int d\mathbf{k} \left(H_{\phi\psi}(k;\tau,\tau) + H_{\psi\phi}(k;\tau,\tau) \right) \quad (45)$$

$$\left\langle \mathbf{u}'_{\rm B} \cdot \mathbf{j}'_{\rm B} \right\rangle = \int d\mathbf{k} H_{ub}(k;\tau,\tau)$$
$$= \frac{1}{4} \int d\mathbf{k} \left(H_{\phi\phi}(k;\tau,\tau) - H_{\psi\psi}(k;\tau,\tau) \right) \quad (46)$$

MODELING OF THE RESIDUAL-ENERGY EQUATION

Residual-Energy Model Equation

Using the analytical results for $R_{\rm T}^{\alpha\beta}$ [Eq. (37)], $W_{\rm T}^{\alpha\beta}$ [Eq. (38)], and $\mathbf{\Gamma}$ [Eq. (39)], we construct a model equation for $K_{\rm R}$. Equation (13) can be modeled as

$$\frac{DK_{\rm R}}{Dt} = -R_{\rm T}^{ab} \frac{\partial U^a}{\partial x^b} + W_{\rm T}^{ab} \frac{\partial B^a}{\partial x^b} - \mathbf{\Gamma} \cdot \mathbf{B} - \varepsilon_{\rm R} + \nabla \cdot \mathbf{T}_{\rm R} \quad (47)$$

where

$$R_{\rm T}^{\alpha\beta} = \frac{2}{3} K \delta^{\alpha\beta} - C_{\rm R} \frac{K}{\varepsilon} K_{\rm R} S^{\alpha\beta} + \text{H.R.T.} \quad (48)$$

$$W_{\rm T}^{\alpha\beta} = \frac{2}{3} W \delta^{\alpha\beta} - C_{\rm R} \frac{K}{\varepsilon} K_{\rm R} M^{\alpha\beta} + \text{H.R.T.} \quad (49)$$

$$\mathbf{\Gamma} = r_1 \mathbf{B} + r_2 \nabla \times \mathbf{B} + r_3 \left(\mathbf{\Omega} + 2\mathbf{\Omega}_{\rm F} \right) \quad (50)$$

with

$$r_1 = C_{\rm r1} \frac{\varepsilon}{K^2} K_{\rm R},\ r_2 = C_{\rm r2} \frac{\varepsilon}{K} H_{\rm T},\ r_3 = -C_{\rm r3} \frac{W}{\varepsilon} H_{\rm T} \quad (51)$$

[$\mathbf{\Omega}(= \nabla \times \mathbf{U})$: mean vorticity]. Here, $\varepsilon_{\rm R}$ and $\nabla \cdot \mathbf{T}_{\rm R}$ are the dissipation and transport rates of $K_{\rm R}$, respectively. With the aid of the algebraic and gradient-diffusion approximations, they are modeled as

$$\varepsilon_{\rm R} = C_{\varepsilon \rm R} \frac{\varepsilon}{K} K_{\rm R} \quad (52)$$

$$\mathbf{T}_{\rm R} = \frac{\nu_{\rm K}}{\sigma_{\rm KR}} \nabla K_{\rm R} \quad (53)$$

($C_{\varepsilon \rm R}$, $\sigma_{\rm KR}$: model constants). In Eqs. (48)-(53), ε is the dissipation rate of K, W is the turbulent cross helicity, and $H_{\rm T}$ is the total turbulent MHD helicity. They are defined by

$$\varepsilon \equiv \nu \left\langle \frac{\partial u'^a}{\partial x^b} \frac{\partial u'^a}{\partial x^b} \right\rangle + \lambda \left\langle \frac{\partial b'^a}{\partial x^b} \frac{\partial b'^a}{\partial x^b} \right\rangle \quad (54)$$

$$W \equiv \left\langle \mathbf{u}' \cdot \mathbf{b}' \right\rangle \quad (55)$$

$$H_{\rm T} \equiv \left\langle \mathbf{u}' \cdot \omega' \right\rangle + \left\langle \mathbf{b}' \cdot \mathbf{j}' \right\rangle \quad (56)$$

We have positive model constants $C_{\rm R}$, $C_{\varepsilon \rm R}$, $\sigma_{\rm KR}$, and C_{rn} (n = 1-3). At present, we may estimate some of them as

$$C_{\rm R} = O(10^{-1}),\ C_{\varepsilon \rm R} = O(10^{-1}) \sim O(1),\ \sigma_{\rm KR} \simeq 1 \quad (57)$$

We should estimate and optimize these values including $C_{\rm rn}$ through various applications of the present model to the real-world flow phenomena.

In Eqs. (48) and (49), H.R.T. denotes the helicity-related terms, whose detailed expressions are suppressed here. As a first step to the modeling of $K_{\rm R}$ equation, we consider the simplest possible model. Then we drop the helicity-related terms [H.R.T. of Eqs. (48) and (49) and the second and third terms of Eq. (50)] in this work.

Structure of the Residual-Energy Equation

We substitute Eqs. (48)-(53) with the helicity-related terms dropped into Eq. (47), and have

$$\frac{DK_{\rm R}}{Dt} = C_{\rm R} \frac{K}{\varepsilon} K_{\rm R} \left(\mathbf{S}^2 - \mathbf{M}^2 \right) - C_{r1} \frac{\varepsilon}{K^2} K_{\rm R} \mathbf{B}^2$$
$$- C_{\varepsilon \rm R} \frac{\varepsilon}{K} K_{\rm R} + \nabla \cdot \left(\frac{\nu_{\rm K}}{\sigma_{\rm KR}} \nabla K_{\rm R} \right) \quad (58)$$

This is the residual-energy ($K_{\rm R}$) equation we propose in this work. The first term on the r.h.s. of Eq. (58) is the production term of $K_{\rm R}$. The second term on the r.h.s. of Eq. (58) shows that $|K_{\rm R}|$ decreases in the presence of $\mathbf{B}$. This corresponds to the Alfvén effect which leads to an equipartition of the kinetic and magnetic energies in the presence of a uniform magnetic field.

One of the prominent features of Eq. (58) is that the generation rate of $K_{\rm R}$ is proportional to $K_{\rm R}$ itself as is seen in the r.h.s. of Eq. (58). As this result, if we have an equipartition or $K_{\rm R} = 0$ at the initial time, such a state holds in the later stage of evolution. On the other hand, if a slight deviation from equipartition occurs, such a seed deviation may grow or decay depending on the magnitude of the velocity strain, $\mathbf{S}^2$, in relation to the counterpart of the magnetic strain, $\mathbf{M}^2$. In this sense, the mean-field structures in the velocity and magnetic field determine the evolution of the residual energy.

TOWARDS APPLICATIONS OF THE RESIDUAL-ENERGY EQUATION

As was referred to in Introduction, the importance of the residual energy is featured in the α- or helicity-dynamo estimate and the solar-wind turbulence. We shall remark on these subjects in the context of the residual-energy equation.

Helicity Dynamo

In the presence of the mean magnetic field $\mathbf{B}$, the turbulent kinetic helicity $\langle \mathbf{u}' \cdot \omega' \rangle$ induces the electromotive force due to fluctuations, $\mathbf{E}_{\rm M}(\equiv \langle \mathbf{u}' \times \mathbf{b}' \rangle)$, that is parallel or antiparallel to $\mathbf{B}$. Such a turbulent motive force leads to the mean electric current ($\mathbf{J}$) configuration parallel or antiparallel to the original $\mathbf{B}$. This is the α or helicity effect and the magnetic-field generation mechanism based on this effect is called the α or helicity dynamo. This dynamo, combined with other mechanisms such as the differential rotation, has considered to play a central role in the magnetic-field generation of the Sun, Earth, galaxies, etc. It is known that the α effect does not solely depend on the kinetic helicity and that the correction due to the turbulent current helicity $\langle \mathbf{b}' \cdot \mathbf{j}' \rangle$ should be included in the effect (Pouquet et al., 1976). Namely, it is the difference between the kinetic and current helicities defined by

$$H \equiv - \left\langle \mathbf{u}' \cdot \omega' \right\rangle + \left\langle \mathbf{b}' \cdot \mathbf{j}' \right\rangle \quad (59)$$

that determines the α or helicity effect (Fig. 2).

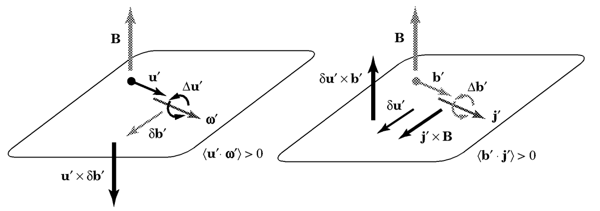

Figure 2: The electromotive force in the residual helicity or α effect: A positive turbulent kinetic helicity ($\langle \mathbf{u}' \cdot \omega' \rangle > 0$) in the presence of the mean magnetic field $\mathbf{B}$ induces the electromotive force antiparallel to $\mathbf{B}$ (Left). On the other hand, a positive turbulent current helicity ($\langle \mathbf{b}' \cdot \mathbf{j}' \rangle > 0$) induces the electromotive force parallel to $\mathbf{B}$ (Right).

The quantity H is called the turbulent MHD residual helicity, whose evolution equation is written as

$$\frac{DH}{Dt} = C_{\rm HR} \frac{K}{\varepsilon} K_{\rm R} \left(M^{ab} \frac{\partial J^b}{\partial x^a} - S^{ab} \frac{\partial \Omega^b}{\partial x^a} \right) - \frac{1}{2} \frac{\partial R^{ab}}{\partial x^a} \Omega^b$$
$$- C_{\rm HB} \frac{\varepsilon^2}{K^3} \mathbf{E}_{\rm M} \cdot \mathbf{B} - C_{\rm H} \frac{K}{\varepsilon} H$$
$$+ \nabla \cdot \left[-\frac{1}{2} (K + K_{\rm R}) \mathbf{\Omega} + \frac{\nu_{\rm K}}{\sigma_{\rm H}} \nabla H \right] \qquad (60)$$

(Yoshizawa, 1998). Here, $\mathbf{R}$ is the Reynolds stress defined by

$$R^{\alpha\beta} \equiv \langle u'^{\alpha} u'^{\beta} - b'^{\alpha} b'^{\beta} \rangle \qquad (61)$$

and $C_{\rm HR}$, $C_{\rm HB}$, $C_{\rm H}$, and $\sigma_{\rm H}$ are model constants. The first term in the r.h.s. shows that the residual helicity can be produced by the nonzero residual energy ($K_{\rm R} \neq 0$). The first part of the transport-rate term [the fifth term in the r.h.s. of Eq. (60)] can be rewritten in a strongly rotating system as

$$P_\Omega = -(\mathbf{\Omega}_{\rm F} \cdot \nabla)(K + K_{\rm R}) \qquad (62)$$

This can be interpreted as the helicity production due to system rotation. If there is an energy inhomogeneity or $\nabla(K + K_{\rm R})$ in the direction of the rotation axis, H is supplied to the system. It is anticipated that P_Ω predominantly produces the residual helicity in a rotating spherical dynamo. We should note that it is not K alone but $K_{\rm R}$ added to K that appears in Eq. (62) and plays a critical role in the helicity generation due to the system rotation. This fact suggests that a reliable estimate of the residual energy is indispensable in obtaining a proper estimate of the helicity generation and consequently of the magnetic-field generation.

Solar Wind

The solar wind is a continuous plasma flow blown away from the Sun with a speed of several hundreds km^{-1}. It carries the solar magnetic field out into the heliosphere. The solar wind and its magnetic field are regarded as a dynamically evolving, inhomogeneous, anisotropic magneto-fluid turbulence (Kallenrode, 2004). Solar-wind turbulence has been extensively investigated by satellite observations. Some of the characteristics of the solar-wind turbulence in the context of the residual energy may be listed as follows (Tu and Marsch, 1995):

(i) *Cross helicity:* High correlations between the velocity and magnetic fluctuations have been ubiquitously observed. A typical value of the normalized cross-helicity, defined by the ratio of the cross-helicity spectrum to the total energy one is near 1 at the heliocentric distance R of 0.3 AU (1 AU $= 1.5 \times 10^{11}$ m);

(ii) *Magnetic dominance:* A systematic deviation from the equipartition between the kinetic and magnetic energies is observed. The Alfvén ratio $r_{\rm A}$, defined by the ratio of the spectral kinetic energy to the magnetic one, is reported to be usually less than unity at the large R (Fig. 3);

(iii) *Radial evolutions:* The values of cross helicity and the Alfvén ratio evolve with the heliocentric distance. The normalized cross helicity evolves from 1 at 0.3 AU to near zero at 20 AU. Near the source surface located at the heliocentric distance of about three times of the solar radius (7×10^8 m), $r_{\rm A} = 1 \sim 1.2$, and $r_{\rm A}$ decreases with R and shows $r_{\rm A} \simeq 0.5$ at $R = 8$ AU without further decline for $R \geq 8$ AU (Fig. 3).

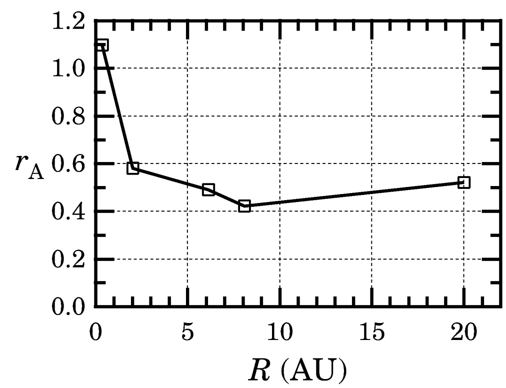

Figure 3: The observed Alfvén ratio $r_{\rm A}$ against the heliocentric distance R: Note that $r_{\rm A}$ is usually less than unity ($r_{\rm A} \simeq 0.5$) in the outer heliosphere ($R > 1$ AU) (Roberts et al, 1990, redrawn).

The magnetic dominance and its tendency in the radial evolution have not been well elucidated by the current MHD theories. We consider this problem from the viewpoint of $K_{\rm R}$ equation.

Typical parameters for the solar wind are $|\mathbf{U}| = 400$ km s^{-1} (low-speed wind) and $|\mathbf{B}| = 5$ nT (which corresponds to 50 km s^{-1} in the Alfvén velocity unit for a plasma with the number density of $n = 5$ cm^{-3}). This indicates that the strain rate of the magnetic field is much smaller than the counterpart of the velocity, $\mathbf{M}^2 \ll \mathbf{S}^2$. Then $K_{\rm R}$ equation is reduced to

$$\frac{\partial K_{\rm R}}{\partial t} = C_{\rm R} \frac{K}{\varepsilon} \mathbf{S}^2 K_{\rm R}$$
$$- \left(C_{\varepsilon \rm R} + C_{r1} \frac{\mathbf{B}^2}{K} \right) \frac{\varepsilon}{K} K_{\rm R} + \nabla \cdot \left(\frac{\nu_{\rm K}}{\sigma_{\rm KR}} \nabla K_{\rm R} \right). \qquad (63)$$

We consider a steady state with a nearly homogeneous $K_{\rm R}$ distribution. In such a state, the production should balance with the dissipation including the Alfvén effect [$\mathbf{B}^2$-related term in Eq. (63)] as

$$C_{\rm R} \frac{K}{\varepsilon} \mathbf{S}^2 K_{\rm R} \simeq \left(C_{\varepsilon \rm R} + C_{r1} \frac{\mathbf{B}^2}{K} \right) \frac{\varepsilon}{K} K_{\rm R} \qquad (64)$$

If we neglect the $\mathbf{B}^2$-related term in Eq. (64), we have an estimate for the ratio of the time scale of turbulence, K/ε, to the time scale of the mean velocity shear, $S^{-1}[S \equiv (\mathbf{S}^2)^{1/2}]$, as

$$\left(\frac{KS}{\varepsilon}\right)^2 \simeq \frac{C_{\varepsilon\mathrm{r}}}{C_\mathrm{R}} = O(1) - O(10^1) \tag{65}$$

[C_R and C_ε are estimated as Eq. (57)].

It is worth noting that the residual energy K_R should be examined simultaneously with the total MHD energy K. The observed Alfvén ratio of $r_\mathrm{A} \simeq 0.5$ corresponds to the $K_\mathrm{R}/K \simeq -0.3$ in terms of these energies. This suggests that the residual energy scaled by the total energy, K_R/K, is the quantity of fundamental importance. The equation for K is written as

$$\frac{DK}{Dt} = -\frac{1}{2}R^{ab}S^{ab} - \mathbf{E}_\mathrm{M}\cdot\mathbf{J} - \varepsilon + \nabla\cdot[\nu_\mathrm{K}(\nabla K)] \tag{66}$$

where the Reynolds stress $\mathbf{R}$ [Eq. (61)] is expressed as

$$R^{\alpha\beta} = \frac{2}{3}K_\mathrm{R}\delta^{\alpha\beta} - \nu_\mathrm{K}S^{\alpha\beta} + \nu_\mathrm{M}M^{\alpha\beta} \tag{67}$$

with

$$\nu_\mathrm{K} = \frac{7}{5}C_\beta\frac{K}{\varepsilon}K, \;\; \nu_\mathrm{M} = \frac{7}{5}C_\gamma\frac{K}{\varepsilon}W \tag{68}$$

(C_β and C_γ: model constants). Combined with K_R equation [Eq. (58)], the equation for the scaled residual energy is derived as

$$\frac{D}{Dt}\frac{K_\mathrm{R}}{K} = G\frac{K_\mathrm{R}}{K}, \tag{69}$$

where G is the growth rate of K_R/K given by

$$\begin{aligned} G &\equiv \frac{1}{K_R}\frac{DK_R}{Dt} - \frac{1}{K}\frac{DK}{Dt} \\ &= C_\gamma\frac{W}{\varepsilon}\left(\frac{7}{10}\mathbf{S}:\mathbf{M} + \mathbf{\Omega}\cdot\mathbf{J}\right) - \cdots \end{aligned} \tag{70}$$

This shows that the initial seed deviation from the equipartition glows or decays depending on the cross helicity W coupled with the velocity and magnetic-field shear rates such as $\mathbf{S}:\mathbf{M}(= S^{ab}M^{ab})$ and $\mathbf{\Omega}\cdot\mathbf{J}$. This situation is related to the fact that K_R is not affected by W while K can be suppressed in the presence of W. In the presence of the cross helicity, the magnitude of the scaled residual energy or $|K_\mathrm{R}/K|$ can be increased owing to the decrease of K even when $|K_\mathrm{R}|$ is not increased.

CONCLUDING REMARKS

With the aid of the TSDIA, expressions for the correlation tensors in the residual-energy equation were derived. Using the results, we proposed a model equation for the residual energy (K_R equation). The structure of K_R equation was examined. It was shown that the evolution of the scaled K_R is determined by the cross helicity (velocity/magnetic-field correlation) of turbulence.

The validity of the K_R-equation model [Eq. (58)] should be examined through applications to the numerical simulation in a solar-wind geometry. For this purpose, we may first consider a solar wind in the equatorial plane and utilize the flow configurations described by the current solar-wind models. In deriving an information on the strain rates, $\mathbf{S}$ and $\mathbf{M}$, we may adopt the magnetic rotator model (Weber and Davis, 1967), one of the most successful models since the pioneering work by Parker (1958) (Figs. 4 and 5).

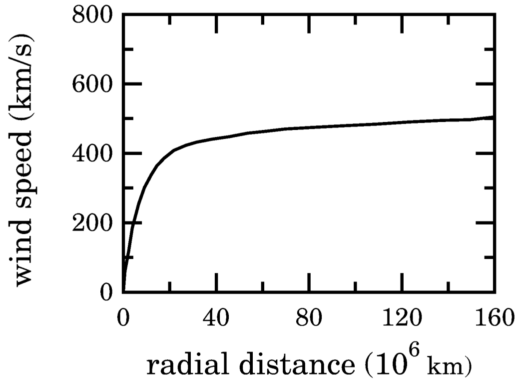

Figure 4: Solar-wind speed against the heliocentric distance: This result was obtained from a thermally-driven-wind model [based on Parker (1958)] and is in good agreement with the observations.

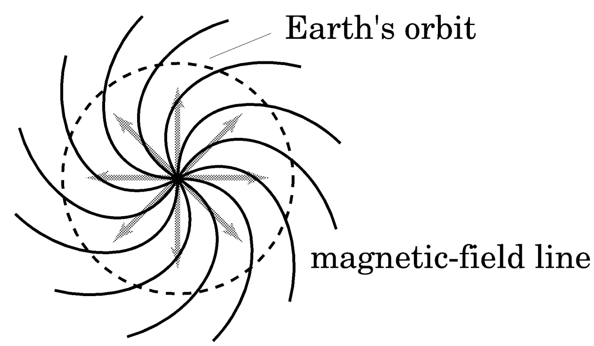

Figure 5: Flow structures from the magnetic rotator model for a solar wind: The magnetic-field lines advected by the solar wind [based on Kallenrode (2004)].

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