MULTI-SCALE ANALYSIS OF CHANNEL TURBULENCE IN WALL-NORMAL DIRECTION BY NONUNIFORM SPLINE WAVELETS ON [-1,1]

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ABSTRACT

In present study, the nonuniform spline wavelets on finite interval are constructed and applied to the analysis of the data from a turbulent channel flow at $Re_\tau = 173$ obtained by direct numerical simulation. The global and local energy spectra in wall-normal direction of Reynolds stress $\langle u'u' \rangle$, $\langle v'v' \rangle$ and $\langle w'w' \rangle$ are obtained, and the results provide us with quantitative scale information in wall-normal direction. The presently constructed wavelets is proved to be an efficient tool for the multiscale decomposition of wall turbulence and it can be further applied to the analysis and simulation of wall-turbulence.

INTRODUCTION

The main difficulty for the understanding and modelling of turbulence is due to its multi-scale nature. The multi-resolution analysis by wavelets has been applied to the investigation of the fascinating turbulence phenomena in recent years, and provides us a plenty of new insights into the multi-scale properties of turbulence. But most of the works in literature is based on the uniformly sampled signals of free turbulence or in the tangential direction of wall turbulence. Due to the lack of an mathematically rigorous and efficient tool, there are only very few studies on the multi-scale properties of wall turbulence in the wall-normal direction, which is of significant importance for the proper understanding and efficient modelling of the near-wall region. To do the multi-scale analysis of wall turbulence in wall-normal direction, the wavelets based on non-uniformly distributed data points on an interval are required. Up to now, the only work is by Frohlich and Uhlmann (2003), in which the orthonormal polynomial wavelet is developed and applied to the analysis of the multi-scale distribution of turbulent kinetic energy in the wall-normal direction of channel turbulence. The nonuniform spline wavelets, which can fit any arbitrary distributed data points, are constructed in present study. The multi-scale distribution of Reynolds stress and transfer of turbulence kinetic energy in the wall-normal direction are investigated using the data base of channel turbulence obtained by direct numerical simulation.

NONUNIFORM SPLINE WAVELETS ON [-1,1]

In the general concept of multi-resolution approximation of $L^2(R)$ proposed by Mallat (1989), we can define the multi-resolution approximation on an interval Ω by constructing the sequence of subspaces $\mathbf{V}_j$ of $L^2(\Omega)$. The subspace $\mathbf{V}_j$ satisfies that $\mathbf{V}_j \subset \mathbf{V}_{j+1}$ and $\mathbf{V}_\infty = L^2(\Omega)$, and is assumed to be spanned by a Riesz basis $\{\phi_{i,j}\}_{i=1}^{n_j}$. Define $\mathbf{W}_{j-1}$ as the L^2-orthogonal complement of $\mathbf{V}_{j-1}$ in $\mathbf{V}_j$, i.e. $\mathbf{V}_j = \mathbf{V}_{j-1} \oplus \mathbf{W}_{j-1}$, and hence we obtain the direct sum decomposition $L^2(\Omega) = \mathbf{V}_0 \oplus \mathbf{W}_0 \oplus \mathbf{W}_1 \oplus \cdots$. The functions in the spaces $\{\mathbf{W}_j\}_j$ are referred to as wavelets and $\{\mathbf{W}_j\}_j$ as wavelet spaces.

Considering the B-splines $B_{i,d,\mathbf{x}}$ defined on the grid sequence $\mathbf{x} = (x_i)_{i=1}^{n+d+1}$, it is of order $d \geq 0$ with the support $[x_i, x_{i+d+1}]$. These B-splines $B_{i,d,\mathbf{x}}$ span a linear space $\mathbf{S}_{d,\mathbf{x}}$. If $\mathbf{y}$ is a subsequence of $\mathbf{x}$ with $\mathbf{y} = (y_i)_{i=1}^{m+d+1}$ $(m < n)$, then we have $\mathbf{S}_{d,\mathbf{y}} \subseteq \mathbf{S}_{d,\mathbf{x}}$. For simplicity, we denote the two sets of B-splines as $\phi_j = B_{j,d,\mathbf{y}}$ $(j = 1, 2 \cdots, m)$ and $\gamma_j = B_{j,d,\mathbf{x}}$ $(j = 1, 2 \cdots, n)$. The linear spaces spanned by the two sets of B-splines are represented by $\mathbf{V}_0 = \mathbf{S}_{d,\mathbf{y}}$ and $\mathbf{V}_1 = \mathbf{S}_{d,\mathbf{x}}$, and the orthogonal complement of $\mathbf{V}_0$ in $\mathbf{V}_1$ is $\mathbf{W}_0$.

Following the method by Lyche and Morken (1992), the spline wavelet basis of minimal support can be constructed. The nonuniform spline wavelet $\boldsymbol{\psi}$ is orthogonal with respect to scales but not to positions. We can construct the dual wavelet basis to obtain the position duality. The dual wavelet basis $\tilde{\boldsymbol{\psi}}$ can be represented by $\tilde{\boldsymbol{\psi}} = \mathbf{A} \cdot \boldsymbol{\psi}$. Let $\left\langle \tilde{\boldsymbol{\psi}}, \boldsymbol{\psi}^{\boldsymbol{T}} \right\rangle = \mathbf{I}$, and we have $\mathbf{A} = \left\langle \boldsymbol{\psi}, \boldsymbol{\psi}^{\boldsymbol{T}} \right\rangle^{-1}$.

In the wavelet space, any function f can be represented as $f = \tilde{\mathbf{d}}^T \boldsymbol{\psi}$ with $\mathbf{d}^T = \tilde{\mathbf{d}}^T \cdot \mathbf{A}^{-1}$. According to the orthogonality and duality of the wavelets, we have

$$\int_\Omega f^2 d\Omega = \sum_{j=0}^{\infty} \mathbf{d}_j^T \cdot \tilde{\mathbf{d}}_j + \mathbf{c}_0^T \cdot \tilde{\mathbf{c}}_0 \tag{1}$$

In present study, the nonuniform spline wavelets on interval have been applied to the analysis of the wall-normal multi-scale properties of channel turbulence. The data is obtained by direct solving the incompressible Navier-Stokes equations with Fourier-Galerkin and Chebyshev-Tau method. The Reynolds number based on channel half width and wall friction velocity is 173. The computational domain in streamwise, wall-normal and spanwise directions is $4\pi \times 2 \times 2\pi$, and corresponding grid system is $128 \times 129 \times 128$. The grid distribution in the wall-normal direction is given by $y_i = \cos(\pi i/N)$ $(i = 0, 1 \cdots, N)$. The spline wavelets based on this grid sequence on [-1,1] are constructed firstly, and the wall-normal multi-scale properties of Reynolds stress are investigated.

Figure 1 shows the constructed wavelets with different scale index j at different position i. It can be seen that wavelets are not translationally invariant. For the same scale index, it has a finer resolution near the wall.

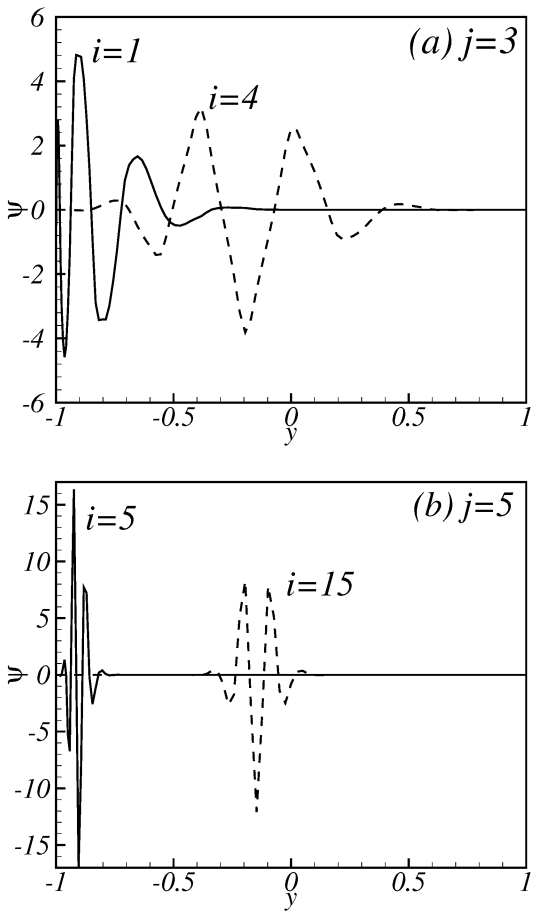

Figure 1: Nonuniform spline wavelets on [-1,1]. (a) $j = 3$, (b) $j = 5$.

WAVELET SPECTRA REPRESENTATION

For a better understanding of the results, we associate a "physical scale" with each wavelet function. Let z^{ψ}_{ji} be the center of wavelet ψ, which scale and position indices are denoted by j and i, and L is the length of the whole interval, the physical scale is

$$s_{ji} = \frac{L}{2}\begin{cases} \frac{z^{\psi}_{ji+1}+z^{\psi}_{ji}}{2}+1, & \text{if} \quad i=0 \\ 1-\frac{z^{\psi}_{ji}+z^{\psi}_{ji-1}}{2}, & \text{if} \quad i=2^j-1 \\ \frac{z^{\psi}_{ji+1}+z^{\psi}_{ji-1}}{2}, & \text{otherwise} \end{cases} \qquad (2)$$

Define the physical wave number as $k_{ji} = 1/s_{ji}$, we get the expression for the global and local energy spectral density

$$E(k_m) = \frac{1}{\Delta k_m} \sum_{j,i/k^L_m<k_{ji}<k^R_m} \tilde{d}_{ji} d_{ji}, \qquad (3)$$

$$E(k_m, y) = \frac{2^j}{\Delta k_m} \tilde{d}_{ji} d_{ji}, \qquad (4)$$

As an example, figure 2 shows the wavelet decomposition in wall-normal direction of the instantaneous streamwise fluctuating velocity. From the figure we can see that the constructed spline wavelets have a much higher resolution near the wall than that near the channel center, this is in accordance with that the characteristic scale of turbulence in the near-wall region is much smaller than that in the outer region. The constructed wavelets are proved to be an efficient tool for the multi-scale decomposition in the wall-normal direction.

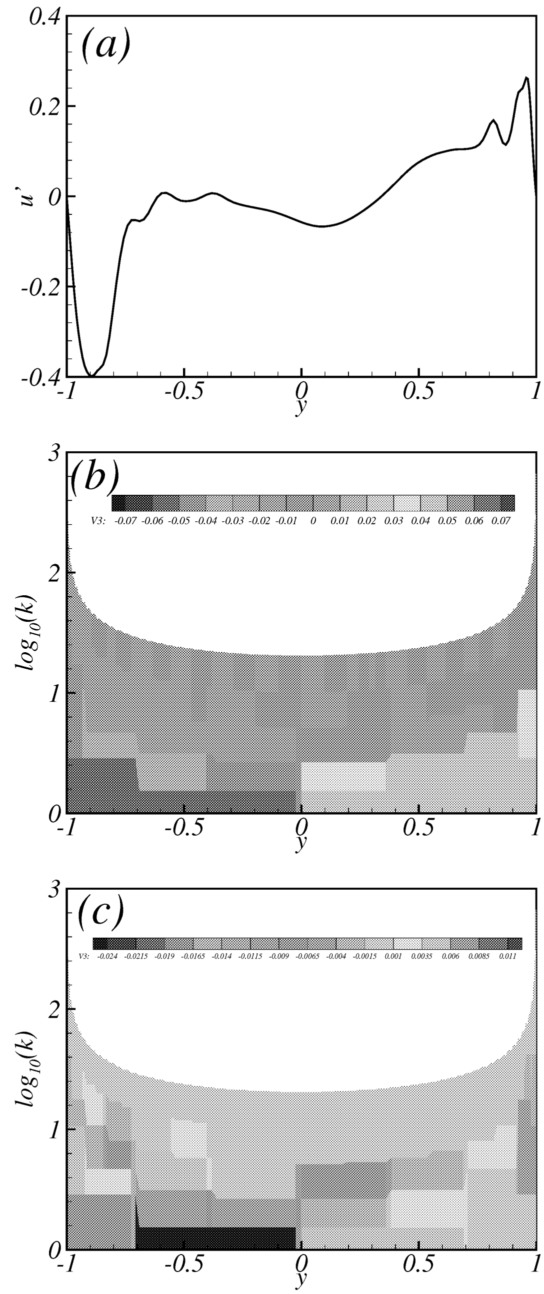

Figure 2: Distribution of (a) original signal, (b) wavelet and (c) dual wavelet coefficients of u'.

MULTISCALE ANALYSIS OF REYNOLDS STRESS IN WALL-NORM DIRECTION

The global and local energy spectra of the Reynolds stress of $\langle u'u' \rangle$, $\langle v'v' \rangle$ and $\langle w'w' \rangle$ are obtained by the method introduced above.

Figure 3 shows the global wall-normal spectra of $\langle u'u' \rangle$,

$\langle v'v' \rangle$ and $\langle w'w' \rangle$. From the figure, we can see that the largest scale possesses most of the energy, which is more than 10 orders of magnitude greater than that in the smallest scales. At all the scales, the energy possessed by $\langle u'u' \rangle$ is greater than that by $\langle v'v' \rangle$ and $\langle w'w' \rangle$. At large scales, $\langle w'w' \rangle$ contains more energy than $\langle v'v' \rangle$, while at smaller scales, $\langle v'v' \rangle$ slightly grater than $\langle w'w' \rangle$.

The local energy spectra of $\langle u'u' \rangle$, $\langle v'v' \rangle$ and $\langle w'w' \rangle$ at $y^+ = 5$, 20 and 66 are shown in figure 4, respectively. From the figures we can see that the distributions of $\langle u'u' \rangle$, $\langle v'v' \rangle$ and $\langle w'w' \rangle$ are similar. In the viscous sublayer ($y^+ = 5$), the energy density decrease monotonically with the increase of the wavenumber. But in the buffer ($y^+ = 20$) and log ($y^+ = 66$) region, there exist the characteristic scale at which the energy spectra reaches maximum value. At $y^+ = 20$, the peak wavenumber is around $10^{0.5} \approx 3.2$, and it is around $10^{0.2} \approx 1.6$ at $y^+ = 66$.

The distributions of $\langle u'u' \rangle$, $\langle v'v' \rangle$ and $\langle w'w' \rangle$ in wall-normal direction with different wavenumber band are shown in figure 5. It can be seen that the energy concentrate near the channel center at large scales, and with the decrease in the scales, the peak position moves much closer to the wall. These results are in accordance with our general knowledge about wall turbulence, and more significantly, provides us with the quantitative information.

CONCLUSION

In present study, the wavelets on finite interval with nonuniform sampling points are constructed by spline functions. This nonuniform spline wavelets can fit any arbitrarily distributed data points. The data from a turbulent channel flow at $Re_\tau = 173$ obtained by direct numerical simulation is analyzed in present paper. The global and local energy spectra in wall-normal direction of Reynolds stress $\langle u'u' \rangle$, $\langle v'v' \rangle$ and $\langle w'w' \rangle$ are obtained, and the results provide us with quantitative scale information in wall-normal direction. The presently constructed wavelets is proved to be an efficient tool for the multiscale decomposition of wall turbulence and it can be further applied to the analysis and simulation of wall-turbulence.

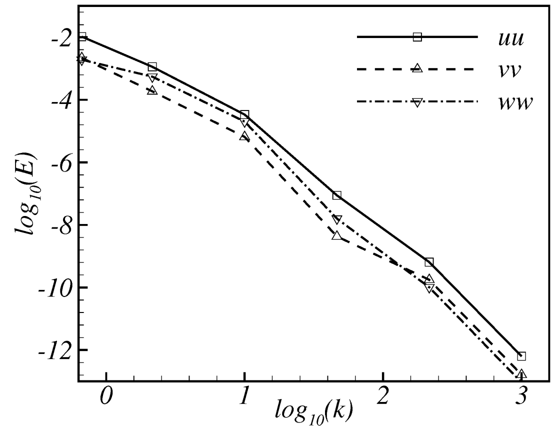

Figure 3: Global energy spectra of $\langle u'u' \rangle$, $\langle v'v' \rangle$,$\langle w'w' \rangle$ and $\langle u'v' \rangle$.

ACKNOWLEDGEMENTS

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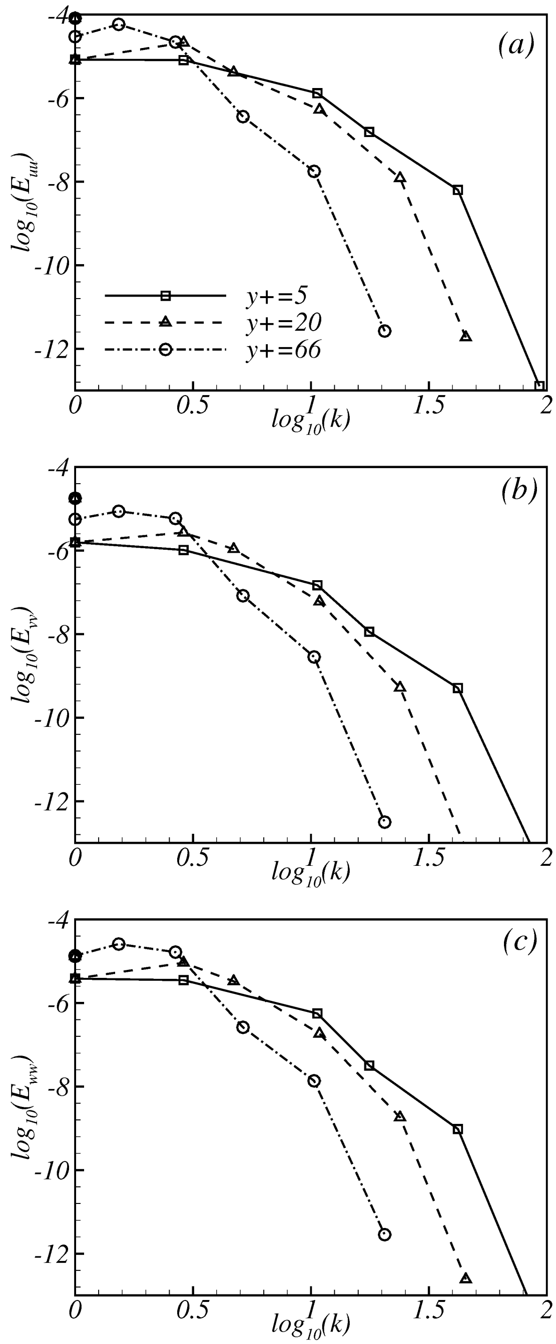

Figure 4: Local energy spectra of (a) $\langle u'u' \rangle$, (b) $\langle v'v' \rangle$ and (c) $\langle w'w' \rangle$ at $y^+ = 5$, 20 and 66, respectively.

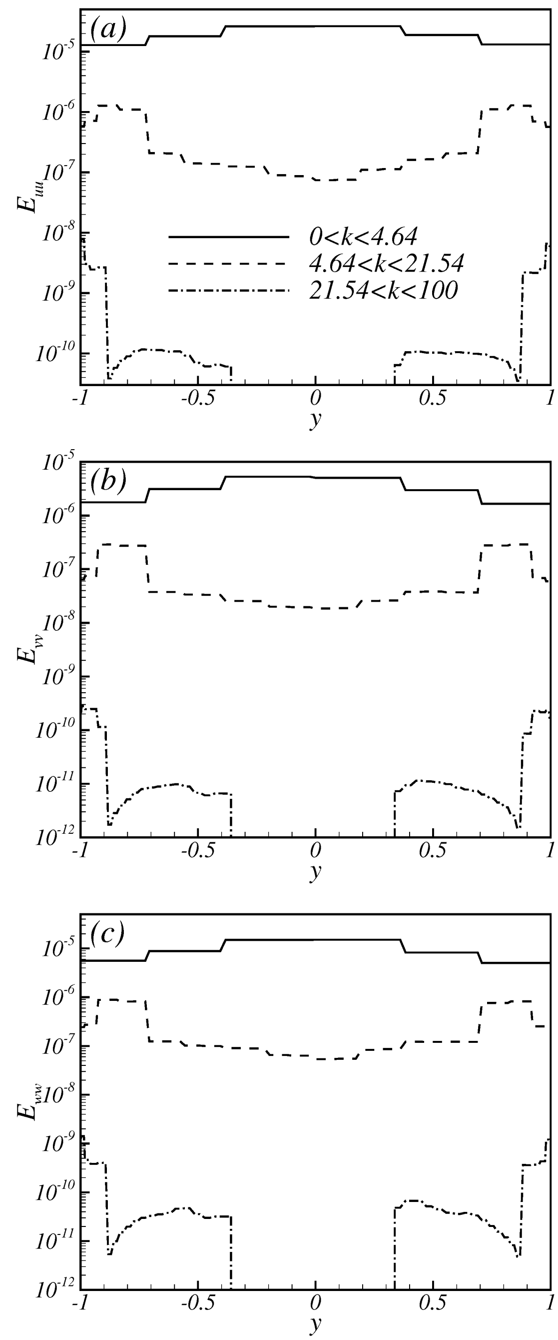

Figure 5: Distribution in wall-normal direction of (a) $\langle u'u' \rangle$, (b) $\langle v'v' \rangle$ and (c) $\langle w'w' \rangle$ for $0 < k < 4.6$, $4.6 < k < 21.5$ and $21.5 < k < 100$.