ON CONSISTENT INVARIANT RANS MODELLING USING NEW SCALING LAWS FROM SYMMETRY METHODS

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ABSTRACT

In RANS modelling it is common practice to use classical canonical flow cases such as the isotropic decay, the logarithmic law of the wall or homogeneous shear flows for calibrating the model constants. With the help of Lie group analysis a broad variety of invariant solutions (scaling laws) can be derived comprising the latter classical solutions as well as a broad variety of new solutions which have so far not been used for model calibration or development. The symmetry methods provide therefore a very useful tool for the improvement of existing turbulence models or may be a guideline for the development of new models. Since the symmetries of fluid motion are admitted by all statistical quantities of turbulent flows as can be taken from the multi-point equations (Oberlack, 2001), we can derive conditions for turbulence models so that they capture the proper flow physics. Concerning these constraints we will exemplary investigate two-equation models as well as Reynolds stress transport models for their capability to reproduce the scaling laws derived from symmetry methods. Therefore the two flow cases of fully developed turbulent rotating pipe flow and a turbulent boundary layer flow have been analyzed.

REQUIRED SYMMETRY CONDITIONS FOR TURBULENCE MODELS

The necessary symmetry conditions for Reynolds-averaged turbulence models have been formulated in Oberlack (2000) as follows:

a.) All symmetries of the two-and multi-point correlation equations have to be admitted by the model equations **(necessary but not sufficient condition!)**.

b.) There should be no additional unphysical symmetries in the model equations also for dimensionally reduced cases such as those admitting rotational symmetry.

c.) The symmetry conditions **(a.)** and **(b.)** have to be admitted by each single model equation and independent of the momentum and continuity equation.

d.) All invariant solutions implied by the symmetries of the two- and multi-point correlation equations also have to be admitted by the model equations.

Condition **(b.)** emerged from a symmetry analyses of the $k-\epsilon$ model in plane and axisymmetric parallel shear flows with rotation. From these testcases Oberlack (2000) found that the $k-\epsilon$ model has too many symmetries which are not contained in the two- and multi- point equations. This obvious shortcoming is further outlined below.

Khor'kova and Verbovetsky (1995) found from a symmetry analysis of the $k-\epsilon$ model that condition **(a.)** is usually fulfilled by the most of modern turbulence model equations. Though the $k-\epsilon$ model apparently admits all necessary symmetries, we find that it is still not capable to reproduce all invariant solutions which are derived from the symmetries of the multi-point equations (condition **(d.)**). A first hint towards this problem is given in Oberlack and Guenther (2003) and Guenther et al. (2004) investigating shear free turbulent diffusion. This clear contradiction may be illuminated by the example of the fully developed rotating pipe flow and the exponential velocity law for the zero-pressure gradient (ZPG) turbulent boundary layer flow.

CONSISTENCY OF RANS MODELS WITH THE ROTATING PIPE FLOW SCALING LAW

Symmetry analysis

In Oberlack (1999) new scaling laws for high-Reynolds-number turbulent pipe flow are derived using symmetry methods. For the analysis an infinite Reynolds-number was assumed and hence viscosity has been neglected. Thus only large-scale quantities such as the mean velocities are determined. Thereby two cases have been distinguished.

The first case is the most general case since no symmetry breaking is imposed on the flow, giving an algebraic scaling law for the axial and azimuthal mean velocity profile. Thus Oberlack (1999) received for the axial mean velocity the velocity defect law

$$\frac{\bar{u}_c - \bar{u}_z}{u_\tau} = \hat{\chi}\left(\frac{\bar{u}_w}{u_\tau}\right)\left(\frac{r}{R}\right)^{\hat{\psi}} \quad , \tag{1}$$

with $\hat{\chi}$ being a function of the velocity ratio $\bar{u}_w/u_\tau$ and since $\bar{u}_w = R\Omega$ therewith dependend on the rotation rate Ω. This dependence has also been derived in Guenther and Oberlack (2005b) analysing second moment closure models.

In Oberlack (1999) it was also found that the algebraic scaling law for the azimuthal velocity component can be rewritten as

$$\frac{\bar{u}_\phi}{\bar{u}_w} = \zeta\left(\frac{r}{R}\right)^{\hat{\psi}} \quad . \tag{2}$$

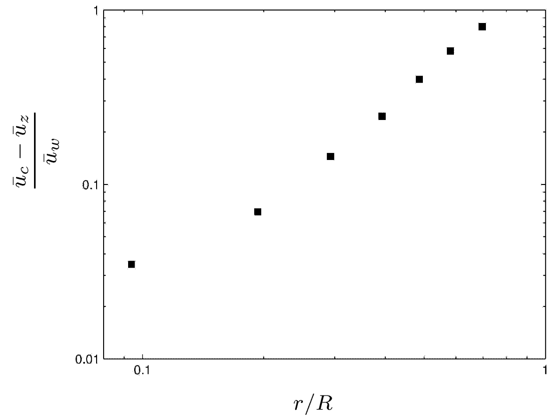

Figure 1: Mean axial velocity $\bar{u}_z$ profile from experiments from Kikuyama et al. (1983) at $Re = 50000$ and $N = 1$

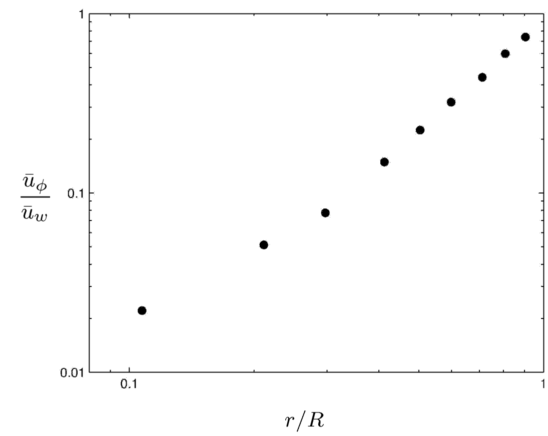

Figure 2: Mean azimuthal velocity profile $\tilde{\bar{u}}_\phi$ from experiments from Kikuyama et al. (1983) at $Re = 50000$ and $N = 1$

The second test case is derived for a special combination of group parameters which applies if an external velocity scale acts on the flow. For this case a logarithmic mean velocity profile for the axial velocity is received in Oberlack (1999) with the singularity appearing on the pipe axis, not at the wall like in the classical law of the wall:

$$\frac{\bar{u}_z}{\bar{u}_w} = \lambda \ln\left(\frac{r}{R}\right) + \omega \quad . \tag{3}$$

Oberlack found that (3) applies in some section of the radius for rapidly rotating pipes in which the wall velocity dominates the friction velocity u_τ and is therefore the symmetry-breaking velocity scale. The corresponding azimuthal velocity is given by

$$\bar{u}_\phi = \gamma \quad , \tag{4}$$

with γ being a constant.

Results from experiments and numerical simulations

The algebraic scaling laws for the azimuthal and axial velocity component (2) is apparent in many experimental and DNS data. The data available in the literature indicate that the algebraic scaling law and its exponent have neither a significant Reynolds number nor a rotation number dependence. Only the extension towards the pipe axis is affected by these two parameters. Using experimental data from Kikuyama et al. (1983) (see figure 1 and 2) Oberlack suggests for the exponents in the scaling laws for the azimuthal (1) and axial (2) velocities $\hat{\psi} \approx 2$ and for the coefficient in (2) $\zeta \approx 1$. (1) and (2) only apply for a moderate rotation number. As the rotation number increases, the rotating wall velocity $\bar{u}_w$ becomes the dominant velocity scale and the axial velocity changes drastically. For the algebraic law for the axial velocity Oberlack found from comparison with DNS data from Orlandi and Fatica (1997) that it is only valid up to $r/R \approx 0.5$. The algebraic law for the azimuthal velocity is valid for $0.3 \leq r/R \leq 0.6$. Below $r/R \approx 0.1$ solid-body rotation is present. These findings are also confirmed in Facciolo (2003). For the region of applicability of the new log law (3) Oberlack suggests $0.5 \leq r/R \leq 0.8$ using data from Orlandi and Fatica (1997). The coefficient λ is negative and approximately equal to -1 and the additive constant ω has been fitted to 0.354. Facciolo (2003) found from his experiments that either the logarithmic region or the value of the coefficient λ differ with the rotation rate. Facciolo performed experiments for the rotation numbers $N = 0.5$, 1.0 and 1.5 and found for $0.5 \leq r/R \leq 0.8$ corresponding to the three rotation numbers three different values for λ being -2.6, -1.5 and -1.1 respectively. Putting λ for all three rotation numbers equal to -1 he found three different regions of fit for the three different rotation numbers, reaching from $0.3 \leq r/R \leq 0.5$ to $0.5 \leq r/R \leq 0.8$

Model implications

Investigating plane and axisymmetric parallel shear flows with rotation Oberlack (2000) found that the standard $k-\epsilon$ model (Hanjalic and Launder, 1976) has too many symmetries (violation of condition **(b.)**), leading to non-physical behavior under certain flow conditions such as rotation or stream-line curvature. This is due to the fact that the $k-\epsilon$ model equations do not contain Coriolis terms for any type of flow so that no symmetry breaking of scaling of time is possible. A complete group analysis of the $k-\epsilon$ equations in cylinder coordinates discloses an additional symmetry of the form:

$$r^* = r\,, \quad \bar{u}_z^* = \bar{u}_z\,, \quad \bar{u}_\phi^* = \bar{u}_\phi + br\,, \quad k^* = k\,, \quad \epsilon^* = \epsilon\,, \tag{5}$$

where b represents the group parameter. This additional symmetry allows to add a solid body rotation to the azimuthal velocity without any change to the remaining flow quantities. Obviously this is unphysical since turbulence is highly sensitive to rotation. The corresponding invariant solutions are

$$\bar{u}_z = \tilde{\chi} r^{k_1} - b_1\,, \quad \bar{u}_\phi = \tilde{\zeta} r^{k_1} - b_2 r \ , \tag{6}$$

$$k = \mu r^{2k_1}\,, \quad \epsilon = \vartheta r^{3k_1 - 1} \ , \tag{7}$$

with k_1, $\tilde{\chi}$, $\tilde{\zeta}$, μ and ϑ being constants. The new unphysical symmetry is characterized by the group parameter b_2. For the azimuthal velocity we receive therewith a combination of a linear and an algebraic law. Insertion of the scaling law into the azimuthal momentum equation shows that only the linear law is a solution and that the algebraic part is just a solution if the group parameter obey special constraints.

Hirai et al. (1988) performed numerical calculations of the present flow case with two two equation and one Reynolds stress model. In their calculations with the standard $k-$

$\epsilon-$model the latter finding of a linear profile for the azimuthal velocity was confirmed.

Furthermore Hirai et al. performed numerical calculations with a modified $k-\epsilon-$model proposed by Launder et al. (1977). In their model Launder et al. introduced a correction of the source term in the dissipation rate equation using the Richardson number R_i. The modified transport equation for the dissipationrate is then

$$0 = \frac{1}{\sigma_\epsilon}\frac{1}{r}\frac{d}{dr}\left(\nu_t r \frac{d\epsilon}{dr}\right) + \tag{8}$$

$$\left[C_{\epsilon 1}\nu_t\left(\left(r\frac{d}{dr}\left(\frac{\bar{u}_\phi}{r}\right)\right)^2 + \left(\frac{d\bar{u}_z}{dr}\right)^2\right) - C_{\epsilon 2}(1-\beta R_i)\epsilon\right]\frac{\epsilon}{k} \quad ,$$

where

$$R_i = \frac{k^2}{\epsilon^2}\frac{\bar{u}_\phi}{r^2}\frac{d}{dr}(r\bar{u}_\phi) \quad . \tag{9}$$

Due to the additional term in the $\epsilon-$equation the unphysical symmetry (5), admitted by the standard $k-\epsilon-$model, is broken.

For their numerical calculations Hirai et al. put $\beta = 0.005$. The calculations give again linear profiles for the azimuthal velocity component which is in contradiction to the algebraic profiles, received from experiments. Though the unphysical symmetry (5) is broken no improvement with respect to the azimuthal velocity is visible. The calculations with the modified model show an increased axial velocity $\bar{u}_z$ near the centerline with increasing swirl strength. The predicted profile of $\bar{u}_z$ becomes rectilinear and hence of cone shape when the swirl is sufficient strong. As a result not even a qualitative agreement to the experimental results is given. This is due to the fact that an artificial symmetry breaking of scaling of time is imposed due to equation (9) in the $\epsilon-$equation. The azimuthal momentum equation simplifies for the fully developed rotating pipe flow to

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^3\nu_t\frac{d}{dr}\left(\frac{\bar{u}_\phi}{r}\right)\right) \tag{10}$$

with $\nu_t = C_\mu\frac{k^2}{\epsilon}$. If we introduce the invariant solution (6) / (7) into (10) we receive:

$$0 = 2(k_1-1)(k_1+1)C_\mu\frac{\mu^2}{\nu}\tilde{\zeta} \quad . \tag{11}$$

This equation gives $k_1 = \pm 1$. Here only the positive sign makes sense since the azimuthal velocity increases from the axis to the wall. This constitutes the linear azimuthal velocity profile which the standard, as well as the modified $k-\epsilon-$model give, though because of different reasons. If we now introduce the solutions with $\bar{u}_\phi = \tilde{\zeta}r$ into the model equations the azimuthal- and the axial velocity components completely decouple in the standard $k-\epsilon-$model. In the modified model a coupling is received due to the additional term which contains the Richardson number. The Richardson number is then given by

$$R_i = 2\frac{\mu^2}{\vartheta^2}\tilde{\zeta}^2 r^{(-2k_1+2)} \quad . \tag{12}$$

In order to be consistent with the scaling law (1)/(2) R_i has to be independent of r which results to the fact that k_1 must equal 1. Therefore we receive a rectilinear profile for the axial profile which becomes more extended for higher rotation rates, since then the influence of the last term in (8) increases. Thus condition **(a.)** is violated due to the structure of the $\epsilon-$equation.

Since experiments and numerical simulations reproduce the scaling laws very good it should be demanded from the model equations to be in accordance with the invariant solutions as well (condition **(d.)**). Therefore we implemented the invariant solutions (1)/(2) into the model equations, corresponding to the procedure described in Guenther and Oberlack (2005a). Thereby we omitted the unphysical symmetry of the $k-\epsilon-$model. Due to the invariant solutions any r dependence cancels out of the equations and a set of algebraic equations is retained connecting the model coefficients as well as the constants received from the symmetry analysis. Introducing no model for the production term the following solutions for μ^3/ϑ^2 and $\mathcal{P}/\vartheta$ can be derived from the simplified equations:

$$\frac{\mu^3}{\vartheta^2} = -\frac{C_{\epsilon 2}-C_{\epsilon 1}}{2C_\mu k_1\left(C_{\epsilon 1}C_k(3k_1+1)+2\frac{1}{\sigma_\epsilon}(1-3k_1)\right)} \quad , \tag{13}$$

$$\frac{\mathcal{P}}{\vartheta} = \frac{2\frac{1}{\sigma_\epsilon}(1-3k_1)+C_kC_{\epsilon 2}(1+3k_1)}{2\frac{1}{\sigma_\epsilon}(1-3k_1)+C_kC_{\epsilon 1}(1+3k_1)} \quad . \tag{14}$$

With the standard model constants and $k_1 \approx 2$, which is suggested by experiments the solution for μ^3/ϑ^2 is positive. Since $\mathcal{P}/\vartheta$ becomes negative for the standard model parameter we can derive the condition

$$\sigma_\epsilon > \frac{2(3k_1-1)}{C_kC_{\epsilon 1}(1+3k_1)} \approx 0.992 \tag{15}$$

under which (14) becomes positive.

For the second case given when an external velocity scale acts on the flow we receive the invariant solutions

$$\bar{u}_z = \sigma + k_2\ln r\,, \quad \bar{u}_\phi = \varsigma + k_3 r \quad , \tag{16}$$

$$k = \varphi\,, \quad \epsilon = \frac{\theta}{r} \quad , \tag{17}$$

with k_2 and k_3 being constant. For this case the invariant solution for the azimuthal velocity (see 16) is once more simplified to $\bar{u}_\phi = \varsigma$. These invariant solutions satisfy the momentum equations insofar that one receives for the azimuthal momentum equation $0 = 0$ and from the axial momentum equation a constant pressure gradient

$$\frac{\partial p}{\partial z} = C_\mu k_1\frac{\varphi^2}{\theta} \tag{18}$$

is obtained. The reduced model equations can be solved for θ^2/φ^2 giving:

$$\frac{\theta^2}{\varphi^2} = C_\mu(\varsigma^2+k_2^2) \text{ and } \frac{\theta^2}{\varphi^2} = C_\mu\frac{C_{\epsilon 1}}{C_{\epsilon 2}}(\varsigma^2+k_2^2) \quad . \tag{19}$$

Here the solution from the $k-$equation is in contradiction to the solution from the $\epsilon-$equation since $C_{\epsilon 1} \neq C_{\epsilon 2}$. Thus it is found that the $k-\epsilon-$model is not in accordance with the invariant solutions (3)/(4) derived from symmetry methods.

Model improvements

In recent years many nonlinear stress-strain relations have been proposed to extend the applicability of linear eddy-viscosity models at modest computational costs. Craft et al. (1996) as well as Shih et al. (1997) applied a nonlinear eddy viscosity model to the rotating pipe flow with reasonable results. The basic assumption behind a nonlinear eddy viscosity model is that the Reynolds stresses are not only related to the rate of strain but also to the rate of rotation. The two scaling parameters k and ϵ are usually used to normalise the Reynolds stresses the rate of strain as well as the rate of rotation as follows:

$$b_{ij} = \frac{\overline{u_i' u_j'}}{2k} - \frac{1}{3}\delta_{ij} \ , \tag{20}$$

$$s_{ij} = \frac{1}{2}\frac{k}{\epsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) , \quad \omega_{ij} = \frac{1}{2}\frac{k}{\epsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{21}$$

Owing to the Cayley-Hamilton theorem, the number of independent invariants and linearly dependent second order tensors that may be formed from the strain- and rotation tensor is finite. In general form the relationship may be written as a tensor polynomial:

$$\boldsymbol{b} = \sum_{\lambda=1}^{10} G_\lambda \boldsymbol{T}^\lambda \ . \tag{22}$$

Thereby the coefficients G_λ are functions of a finite number of scalar invariants. In the general, three-dimensional case there are ten tensors (see e.g. Pope, 1975):

$$\begin{aligned}
\boldsymbol{T}^1 &= \boldsymbol{s}, \\
\boldsymbol{T}^2 &= \boldsymbol{s}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{s}, \\
\boldsymbol{T}^3 &= \boldsymbol{s}^2 - \tfrac{1}{3}\boldsymbol{\delta}\{\boldsymbol{s}^2\}, \\
\boldsymbol{T}^4 &= \boldsymbol{\omega}^2 - \tfrac{1}{3}\boldsymbol{\delta}\{\boldsymbol{\omega}^2\}, \\
\boldsymbol{T}^5 &= \boldsymbol{\omega}\boldsymbol{s}^2 - \boldsymbol{s}^2\boldsymbol{\omega}, \\
\boldsymbol{T}^6 &= \boldsymbol{\omega}^2\boldsymbol{s} + \boldsymbol{s}\boldsymbol{\omega}^2 - \tfrac{2}{3}\boldsymbol{\delta}\{\boldsymbol{s}\boldsymbol{\omega}^2\}, \\
\boldsymbol{T}^7 &= \boldsymbol{\omega}\boldsymbol{s}\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\boldsymbol{s}\boldsymbol{\omega}, \\
\boldsymbol{T}^8 &= \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{s}^2 - \boldsymbol{s}^2\boldsymbol{\omega}\boldsymbol{s}, \\
\boldsymbol{T}^9 &= \boldsymbol{\omega}^2\boldsymbol{s}^2 + \boldsymbol{s}^2\boldsymbol{\omega}^2 - \tfrac{2}{3}\boldsymbol{\delta}\{\boldsymbol{s}^2\boldsymbol{\omega}^2\}, \\
\boldsymbol{T}^{10} &= \boldsymbol{\omega}\boldsymbol{s}^2\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\boldsymbol{s}^2\boldsymbol{\omega}
\end{aligned} \tag{23}$$

and five scalar invariants:

$$I_1 = \{\boldsymbol{s}^2\}, \ I_2 = \{\boldsymbol{\omega}^2\}, \tag{24}$$
$$I_3 = \{\boldsymbol{s}^3\}, \ I_4 = \{\boldsymbol{\omega}^2\boldsymbol{s}\}, \ I_5 = \{\boldsymbol{\omega}^2\boldsymbol{s}^2\} \ .$$

Analyzing the scaling laws (1) and (2) for the fully developed turbulent pipe flow it has been found that only the three tensors $\boldsymbol{T}_5$, $\boldsymbol{T}_6$, $\boldsymbol{T}_{10}$ and the invariant I_2 are sensitive to rotation and hence do not admit the additional unphysical symmetry (5) which is admitted by the standart $k-\epsilon-$model. This founding suggests a new model for the eddy viscosity which incloses the invariant I_2 due to which the additional, unphysical symmetry is broken. From dimensional arguments we derive the simple model

$$\nu_t = C_\mu^* \frac{\epsilon}{I_2} = C_\mu^* \frac{\epsilon}{\{\boldsymbol{\omega}^2\}} \ , \tag{25}$$

which is just one example to model the eddy viscosity in terms of I_2. C_μ^* is thereby the model constant not related to C_μ. Interesting enough using (25) in the $k-\epsilon-$modell for the production term while keeping the standard $\nu_t = C_\mu \frac{k^2}{\epsilon}$ for the diffusion term results in an additional scaling symmetry. Thus we receive three scaling symmetries given by:

$$\begin{aligned}
G_{s1}&: r^* = e^{c_1} r\,,\ \bar{u}_z^* = \bar{u}_z\,,\ \bar{u}_\phi^* = \bar{u}_\phi\,,\ k^* = k\,,\ \epsilon^* = e^{-c_1}\epsilon\,,\\
G_{s2}&: r^* = r\,,\ \bar{u}_z^* = \bar{u}_z\,,\ \bar{u}_\phi^* = \bar{u}_\phi\,,\ k^* = e^{2c_2} k\,,\ \epsilon^* = e^{3c_2}\epsilon\,,\\
G_{s3}&: r^* = r\,,\ \bar{u}_z^* = e^{c_3}\bar{u}_z\,,\ \bar{u}_\phi^* = e^{c_3}\bar{u}_\phi\,,\ k^* = k\,,\ \epsilon^* = \epsilon\,,
\end{aligned} \tag{26}$$

where the c_i represent the group parameters. Due to this additional symmetry the exponents in the algebraic laws for k respectively ϵ and the velocity components decouple from each other. Whether this has a physical meaning is unclear at this point. Still an equivalent finding has been mentioned in Khujadze and Oberlack (2004) investigating the two point correlation equation for the zero-pressure gradient (ZPG) turbulent boundary layer flow.

CONSISTENCY OF RANS MODELS WITH THE EXPONENTIAL VELOCITY DEFECT LAW

Symmetry analysis

Analyzing the multi-point-correlation equations for parallel turbulent shear flows, and ZPG turbulent boundary layer flows a new exponential law has been found in Oberlack (2001) which was identified as an explicit analytic form of the velocity defect law. Scaling the wall-normal coordinate in the outer region with the Clauser-Rotta length scale ($\Delta = \frac{\delta_* \bar{u}_\infty}{u_\tau}$, where δ_* is the displacement thickness) and applying the free stream boundary condition the exponential law can be written in general form as

$$\frac{\bar{u}_\infty - \bar{u}_1}{u_\tau} = F(\eta) = \alpha \exp(-\beta\eta)\,, \tag{27}$$

where α and β are universal constants and $\eta = y/\Delta$. In Oberlack (2001) the exponential law (27) was solely derived from the Lie symmetries of the Navier-Stokes and in turn from the multi-point-correlation equations introducing the assumption of symmetry breaking of the scaling of space.

Results from experiments and numerical simulations

Recently the theory has been carefully tested against very high quality experimental data from the KTH database (Lindgren et al., 2004) and the Illinois windtunnel data (Nagib, 2004). It can be shown that the exponential law fits the experimental data very well in the range of about $0.03 \le y/\Delta \le 0.10$ as can be seen from figure 3. The constants are determined to $\alpha = 10.6$ and $\beta = 9.34$.

Figure 4 shows the results of a direct numerical simulation (DNS) of a ZPG boundary layer performed for $R_\theta = 2240$ (see Khujadze and Oberlack, 2004). The DNS results show an exponential law in the region $0.025 \le y/\Delta \le 0.15$.

Model implications

Since a very good agreement between theory, experiments and numerical simulations is observed it should also be demanded from any RANS models to be in accordance with Oberlack (2001). Thus a further symmetry analysis of the $k-\epsilon$ model for ZPG boundary layer flows has been performed. Using this as well as the symmetry breaking of the scaling of space we obtain besides (27) the following set of invariant solutions:

$$k = C\exp(-2\beta\eta)\,,\ \epsilon = D\exp(-3\beta\eta)\,, \tag{28}$$

where C and D are assumed to be universal parameter. The model equations thus formally admit all symmetries of the correlation equations (condition **(a.)**). Therefore it remains

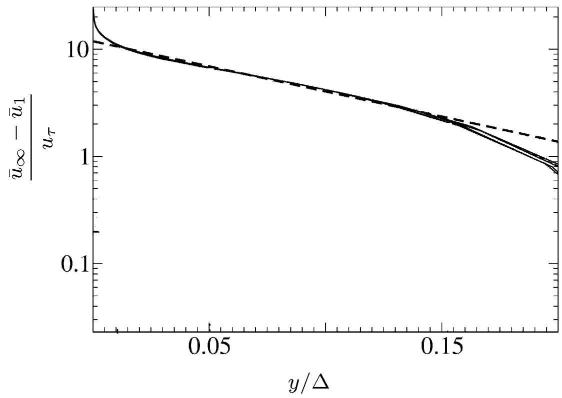

Figure 3: ZPG boundary layer mean velocity profiles from experiments at different Reynolds numbers, performed at KTH (Stockholm) (Lindgren et al., 2004); - - - exponential law (Oberlack, 2001).

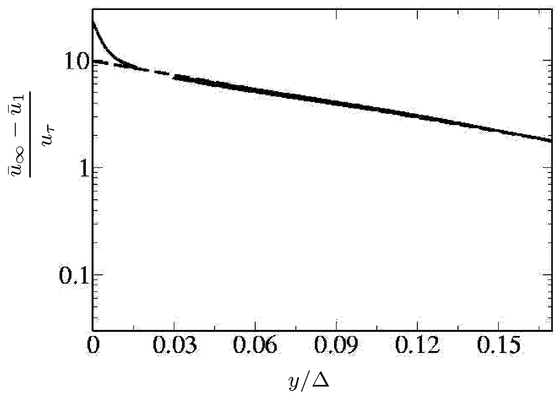

Figure 4: Mean velocity profiles from DNS at $Re_\theta = 2240$, (Khujadze and Oberlack, 2004); - - - exponential law (Oberlack, 2001).

to check if they also admit the invariant solutions (condition **(d.)**). The models which have been tested concerning this requirement are the one-equation model from Spalart and Allmaras (1992), the classical $k-\epsilon$ model from Hanjalić and Launder (1976), the $k-\omega$ model from Wilcox (1993), the $v^2 f$ model from Durbin (1991), the *SST* model from Menter (1994), the $\sqrt{k}L$ model from Menter and Egorov (2004), the $k-kL$ model from Rotta (1968) and as an example of a Reynold stress model the *LRR* model from Launder et al. (1975). Thereby it was found that the model constants of all these models are in contradiction to the theory.

In the following we will examplary point out the problems appearing in the *Spalart-Almaras*, the $k-\epsilon$- and the *LRR* model. For our investigations we assumed that the exponential region is characterized by an equilibrium between production, diffusion and dissipation (see Fig. 5). All statistical quantities depend only on the wallnormal coordinate leading to a simplification of the model equations. Introducing the invariant solutions (27) and (28) into these simplified model equations we find that any exponential dependence on x_2 cancels out. Hence a set of algebraic equations is retained connecting the model coefficients as well as α and β. Interesting enough solving these equations we obtain coefficients for the exponential law which are in contradiction to common model values.
Subsequently, we discuss in some detail three of the above mentioned model equations. Replacing the dependent variable $\tilde{\nu}$ in the *Spalart-Almaras* model by the invariant solution

$$\tilde{\nu} = E \exp(-\beta y) \qquad (29)$$

and neglecting molecular viscosity we receive the coefficient

$$E = -\frac{C_{b1}\sigma\alpha u_\tau \Delta}{2\beta(2+C_{b2})}\,, \qquad (30)$$

whereby C_{b1}, σ and C_{b2} are model constants. We can thus derive the condition

$$\frac{C_{b1}\sigma}{2(2+C_{b2})} < 0\,, \qquad (31)$$

under which a proper modelling of the exponential region is assured. Since C_{b1}, σ and C_{b2} are positive, changing the algebraic sign of one of the coefficients would probably lead to a deficient modelling if other flow cases are considered.
The same problem appears if the invariant solutions (27) and (28) are introduced into the $k-\epsilon$ model. Here we receive the following dependence of the coefficients on the model constants:

$$C = \frac{\sigma_k\sigma_\epsilon(C_{\epsilon1}-C_{\epsilon2})}{6C_{\epsilon2}\sigma_\epsilon-12\sigma_k}\alpha^2 u_\tau^2\,, \qquad (32)$$

$$D = \sqrt{\frac{C_\mu\sigma_k^2\sigma_\epsilon^2(C_{\epsilon1}-C_{\epsilon2})^2\left(1+\frac{\sigma_\epsilon(C_{\epsilon1}-C_{\epsilon2})}{C_{\epsilon2}\sigma_\epsilon-2\sigma_k}\right)}{(6C_{\epsilon2}\sigma_\epsilon-12\sigma_k)^2}\frac{\alpha^3 u_\tau^3\beta}{\Delta}}\,, \qquad (33)$$

leading with the standard model constants to

$$C = -0.21\alpha^2 u_\tau^2\,, \quad D = 0.03\sqrt{-1}\frac{\alpha^3 u_\tau^3\beta}{\Delta} \qquad (34)$$

which apparently is unphysical. From equation (32) a condition for the model constants of the $k-\epsilon$ model may be derived, guaranteeing a proper modeling of the exponential velocity law:

$$C_{\epsilon1} < C_{\epsilon2} \quad , \frac{C_{\epsilon2}\sigma_\epsilon}{\sigma_k} < 2\,. \qquad (35)$$

Thus e.g. changing σ_ϵ to 1.04 which comes from the the second condition (and is also in accordance with (15)) and keeping the numerical value of the other model constants would give positive values for C and D. Since the model constants are related by

$$\kappa^2 = \sigma_\epsilon C_\mu^{1/2}(C_{\epsilon2}-C_{\epsilon1}) \qquad (36)$$

due to the logarithmic law of the wall a change of σ_ϵ to 1.04 demands an adjustment of the Karman constant to 0.38. In the literature the values for κ range from 0.38 to 0.43. Therefore $\kappa = 0.38$ is an acceptable choice and guarantees together with $\sigma_\epsilon = 1.04$ a correct reproduction of both the log-law region and the exponential law.
For the *LRR* model such a discrepancy appears in the model constants that all coefficients of the scaling laws become zero if the invariant solutions are introduced into the model equations. The reason for these mismatches seem to be due to the fact that the given models are all calibrated employing the classical flow cases. A calibration of the models using symmetry methods would probably improve the described shortcomings.

CONCLUSIONS

Investigating a fully developed turbulent rotating pipe flow as well as ZPG flow it could be shown that for a proper modelling it is not sufficient that the model equations admit all

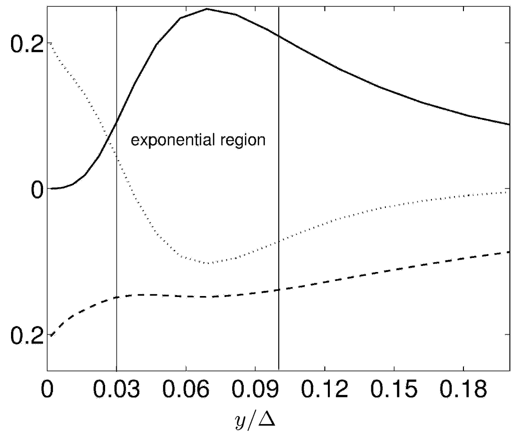

Figure 5: The turbulent-kinetic-energy budget in a turbulent boundary layer at $Re_\theta = 1410$ (Spalart, 1988); —— production, - - - dissipation, ··· diffusion.

symmetries of the two- and multi-point correlation equations. We derived therefore as further condition for RANS models that there should be no additional unphysical symmetries in the model equations and that the scaling laws which are derived from the symmetries have to be reproduced by the models.

From the given examples we can further propose conditions for the model constants which have to be fulfilled so that the model equations admit the invariant solutions.

It could thus be shown that a considerable improvement in the field of turbulence modeling may be received if symmetry methods are used for the calibration or development of turbulence models.

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