SPECTRAL EVOLUTION OF TURBULENCE IN THE LIMIT OF SLOW VARIATION

Robert Rubinstein
Computational AeroSciences Branch,
NASA Langley Research Center
Hampton, Virginia 23681-2199, USA
r.rubinstein@larc.nasa.gov

Stephen L. Woodruff
Center for Advanced Propulsion Systems,
Florida State University
Tallahassee, Florida 32306-4120, USA
slw@caps.fsu.edu

ABSTRACT

The ϵ equation is considered from the viewpoint of spectral evolution in a closure theory. The balance of vortex stretching and enstrophy destruction, on which the ϵ equation depends, occurs whenever the spectrum can be described by parameters that vary slowly relative to the turbulence itself. The central question for modeling is whether these parameters satisfy closed equations of motion. A multiple scale analysis of a closure model in the slow variation limit, analogous to the Chapman-Enskog expansion of kinetic theory, suggests that a universal ϵ equation does not exist. Further evidence against this possibility is given by constructing self-similar states of turbulence evolution each of which is consistent with an ϵ equation with different constants.

INTRODUCTION

The ϵ equation is often considered a weak link in turbulence modeling. Single-point models begin with exact equations for certain basic correlations; they assert that the unknown correlations that these equations inevitably contain can be expressed in terms of these basic correlations. Typically, no statistical or other theory underlies this assertion, which is justified after the fact by urging that it works well in computations.

The ϵ equation presents a much more formidable challenge: the classic analysis of the problem by Tennekes and Lumley (1972) demonstrates that formulating an ϵ equation requires modeling the $O(Re^0)$ *difference* between two correlations each of which is $O(Re^{1/2})$. From this viewpoint, no definite correlation is even being modeled. For this reason, the ϵ equation is typically treated in the modeling literature as unavoidable, but despite its many successful applications, fundamentally questionable.

The ϵ equation will be studied here in what appears to be the simplest possible context: homogeneous isotropic turbulence driven by a statistically unsteady random force. This problem will be analyzed using spectral closure. We will investigate the conditions under which the predictions of closure can be approximated by the solutions of a two-equation model.

Whereas a closure gives a definite equation for the evolution of ϵ which is obtained by simply integrating the closure equation itself, two-equation modeling requires that this equation be expressed in terms of ϵ itself, the kinetic energy k, and the total production P. The possibility of this reformulation is the subject of this paper. Even if the energy spectrum can be described well by two scalar parameters, it is far from evident that these parameters satisfy closed equations of motion.

It might be objected that our analysis replaces the original problem of modeling the statistical Navier-Stokes equations by the problem of modeling a spectral closure. But in fact, no heuristic used in modeling does not apply to the spectral closure we will use; we can therefore argue that a good model for the ϵ equation of statistical Navier-Stokes theory should also be a good model for our simpler problem. This replacement problem also has the important advantage that definite answers can be obtained independently of CFD calculations.

For simplicity, the analysis will be based on the Heisenberg model. Of the many approaches to linking two-point and single-point models, we follow the parametrization of the spectrum by two scalars introduced by Besnard *et al* (1996). There is an analogy between their *ansatz* and the *normal solution* adopted as a starting point for the Chapman-Enskog expansion of kinetic theory. It may be useful to think of the Navier-Stokes equations themselves as a five-equation model for the Boltzmann equation.

The conclusions are that the balance between vortex stretching and enstrophy destruction required by the ϵ equation can indeed be justified in a very large class of problems in which the spectrum changes slowly. Nevertheless, a closed and universal ϵ equation cannot be derived.

THE HEISENBERG CLOSURE

In the Heisenberg closure, the spectral evolution equation is

$$\dot{E}(\kappa,t) = P(\kappa,t) - \frac{\partial}{\partial\kappa}\mathcal{F}[E(\kappa,t)] - D(\kappa,t) \tag{1}$$

where $E(\kappa,t)$ is the energy spectrum, the dissipation spectrum is $D(\kappa,t) = 2\nu\kappa^2 E(\kappa)$, and the spectral energy flux $\mathcal{F}$ is the functional of E

$$\mathcal{F}[E(\kappa,t)] = c\int_0^\kappa d\kappa'\ (\kappa')^2 E(\kappa',t)\int_\kappa^\infty dp\ \sqrt{\frac{E(p,t)}{p^3}} \quad (2)$$

Single point moments are defined as usual by

$$\begin{aligned} k(t) &= \int_0^\infty d\kappa\ E(\kappa,t) \\ \epsilon(t) &= \int_0^\infty d\kappa\ 2\nu\kappa^2 E(\kappa,t) \\ P(t) &= \int_0^\infty d\kappa\ P(\kappa,t) \end{aligned} \quad (3)$$

the total kinetic energy, dissipation rate, and total production, respectively.

Suppose first that the production spectrum is independent of time. Then if there exists an inertial range of scales in which $P(\kappa) \approx 0$ and $D(\kappa) \approx 0$, the steady solution of Eq. (1) is the Kolmogorov spectrum

$$E(\kappa) = C_K \epsilon^{2/3}\kappa^{-5/3} \quad (4)$$

characterized by a constant energy flux

$$\mathcal{F}[E(\kappa)] = P = \epsilon \quad (5)$$

More generally, the steady inviscid solution has the form

$$E(\kappa) = C_K \epsilon^{2/3}\kappa^{-5/3} f(\kappa/\kappa_0) \quad (6)$$

in terms of a function f which satisfies $f(x) \approx 1$ when $x \gg 1$.

Integrating Eq. (1) over all wavenumbers gives the energy balance

$$\dot{k} = P - \epsilon \quad (7)$$

Multiplying Eq. (1) by $2\nu\kappa^2$ and integrating gives the exact dissipation rate equation

$$\dot{\epsilon} = \Pi - S - G \quad (8)$$

where

$$\begin{aligned} \Pi(t) &= \int_0^\infty d\kappa\ 2\nu\kappa^2 P(\kappa,t) \quad (9) \\ S(t) &= \int_0^\infty d\kappa\ 2\nu\kappa^2 \frac{\partial}{\partial\kappa}\mathcal{F}[E(\kappa,t)] \quad (10) \\ G(t) &= \int_0^\infty d\kappa\ 4\nu^2\kappa^4 E(\kappa,t) \quad (11) \end{aligned}$$

TIME-DEPENDENT SOLUTIONS

Suppose that the production spectrum depends on two time-dependent scalars: the peak forcing scale λ_0 and the total production amplitude P, so that

$$P(\kappa,t) = P(t)\pi(\kappa/\lambda_0(t)) \quad (12)$$

This form of the production spectrum naturally suggests looking for a general time dependent solution of Eq. (1) of the form

$$E_0(\kappa,t) = C_K\epsilon(t)^{2/3}\kappa^{-5/3}\phi(\kappa/\kappa_0(t)) \quad (13)$$

a local Kolmogorov solution analogous to the *normal solution* in kinetic theory which coincides with the spectral *ansatz* of Besnard *et al* (1996).

Because the integral operator which determines the flux through Eq. (2) does not depend on time, it is evident that

$$\frac{\partial}{\partial\kappa}\mathcal{F}[E_0(\kappa,t)] = 0 \quad (14)$$

when $P(\kappa,t), D(\kappa,t) \approx 0$. However, $E_0(\kappa,t)$ does not satisfy Eq. (1) because its time derivative

$$\dot{E}_0(\kappa,t) = \left[\frac{2}{3}\frac{\dot{\epsilon}}{\epsilon} - \frac{\phi'}{\phi}\frac{\kappa}{\kappa_0}\frac{\dot{\kappa_0}}{\kappa_0}\right] E_0(\kappa,t) \quad (15)$$

need not vanish. A solution of Eq. (1) should therefore be sought as an expansion

$$E(\kappa,t) = E_0(\kappa,t) + E_1(\kappa,t) + \cdots \quad (16)$$

Following the Chapman-Enskog expansion, regroup the terms in the expansion so that the lowest order correction term $E_1(\kappa,t)$ satisfies the integral equation

$$\dot{E}_0(\kappa,t) - P(\kappa,t) + \frac{\partial}{\partial\kappa}\mathcal{F}[E_0(\kappa,t)] + D(\kappa,t) = \mathcal{L}[E_1(\kappa,t)] \quad (17)$$

where $\mathcal{L}$ is the linearized Heisenberg operator defined so that

$$\frac{\partial}{\partial\kappa}\mathcal{F}[E_0+E_1] \approx \frac{\partial}{\partial\kappa}\mathcal{F}[E_0] + \mathcal{L}[E_1] \quad (18)$$

Explicitly,

$$\begin{aligned} \mathcal{L}[E_1(\kappa,t)] = \frac{\partial}{\partial\kappa}\int_0^\kappa d\kappa'(\kappa')^2 E_1(\kappa',t)\ \times \\ \int_\kappa^\infty dp\ \sqrt{\frac{E_0(p,t)}{p^3}} + \frac{1}{2}\frac{\partial}{\partial\kappa}\int_0^\kappa d\kappa'(\kappa')^2 E_0(\kappa',t)\ \times \\ \int_\kappa^\infty dp\ \frac{E_1(p,t)}{\sqrt{E_0(p,t)p^3}} \end{aligned} \quad (19)$$

Assume that the total kinetic energy is contained in the solution E_0, so that $\int_0^\infty d\kappa\ E_0(\kappa,t) = k$ and $\int_0^\infty d\kappa\ E_1(\kappa,t) = 0$ Formally then, E_1 is treated as a higher order term than E_1 and would enter an integral equation for the next term in the expansion, E_2.

The solvability condition for Eq. (17) is

$$\begin{aligned} \int_0^\infty d\kappa\ \{\dot{E}_0(\kappa,t) - P(\kappa,t) + \frac{\partial}{\partial\kappa}\mathcal{F}[E_0(\kappa,t)] \\ +D(\kappa,t)\}\Psi(\kappa) = 0 \end{aligned} \quad (20)$$

where $\Psi(\kappa)$ is any solution of the integral equation

$$\mathcal{L}^\dagger[\Psi(\kappa,t)] = 0 \quad (21)$$

Explicitly,

$$\begin{aligned} \mathcal{L}^\dagger[\Psi(\kappa,t)] = -c\kappa^2\int_\kappa^\infty dq\ \frac{\partial\Psi}{\partial q}\int_q^\infty dp\ \sqrt{\frac{E_0(p,t)}{p^3}} \\ -\frac{c/2}{\sqrt{E_0(\kappa,t)\kappa^3}}\int_0^\kappa dq\ \frac{\partial\Psi}{\partial q}\int_0^q d\kappa'(\kappa')^2 E_0(\kappa',t) \end{aligned} \quad (22)$$

which has the obvious solution $\Psi(q,t) \equiv 1$. Accordingly, the solvability condition becomes simply

$$\int_0^\infty d\kappa \, \{\dot{E}_0(\kappa,t) - P(\kappa,t) + \frac{\partial}{\partial\kappa}\mathcal{F}[E_0(\kappa,t)] + D(\kappa,t)\} = 0 \tag{23}$$

which is equivalent to the energy balance Eq. (7). Thus, Eq. (17) for $E_1(\kappa,t)$ satisfies the required solvability condition.

In the inertial range where $P(\kappa)$ and $D(\kappa)$ are both negligible and $\mathcal{F}[E_0] = 0$, Eq. (17) has the form

$$\left[\frac{2}{3}\frac{\dot{\epsilon}}{\epsilon} - \frac{\phi'}{\phi}\frac{\kappa}{\kappa_0}\frac{\dot{\kappa_0}}{\kappa_0}\right] E_0(\kappa,t) \approx \mathcal{L}[E_1(\kappa,t)] \tag{24}$$

The definition Eq. (19) of $\mathcal{L}$ and the Kolmogorov scaling of E_0 show that Eq. (17) admits the scaling solution

$$E_1(\kappa) \sim \frac{\dot{\epsilon}}{\epsilon}\,\epsilon^{1/3}\kappa^{-7/3} \tag{25}$$

with the $\kappa^{-7/3}$ scaling proposed for time dependent turbulence by Yoshizawa (1994); however, in agreement with Kolmogorov's notions of locality, this correction does not depend on κ_0.

Eq. (25) suggests that the expansion Eq. (14) is an expansion in power of $\dot{\epsilon}/\epsilon^{4/3}\kappa^{-2/3}$, the ratio of the imposed frequency $\dot{\epsilon}/\epsilon$ to the local turbulence frequency $\epsilon^{1/3}\kappa^{2/3}$. This quantity has the role of the Knudsen number in this expansion, which thereby assumes weak time dependence of the spectrum.

THE BALANCE BETWEEN VORTEX STRETCHING AND ENSTROPHY DESTRUCTION

Yoshizawa's correction term Eq. (25) has an important implication for the formulation of the ϵ equation. Because $P(\kappa)$ is concentrated near large scales, it can be neglected in Eq. (8), which reduces to $\dot{\epsilon} = -S - G$. Substituting the expansion Eq. (16) in the definitions Eqs. (10) and (11), $S = S_0 + S_1 + \cdots$ and $G = G_0 + G_1 + \cdots$ where

$$\begin{aligned} S_0 &= \int_0^\infty d\kappa \; 2\nu\kappa^2 \frac{\partial}{\partial\kappa}\mathcal{F}[E_0(\kappa,t)] \\ G_0 &= \int_0^\infty d\kappa \; 4\nu^2\kappa^4 E_0(\kappa) \\ S_1 &= \int_0^\infty d\kappa \; 2\nu\kappa^2 \mathcal{L}[E_1(\kappa,t)] \\ G_1 &= \int_0^\infty d\kappa \; 4\nu^2\kappa^4 E_1(\kappa) \end{aligned} \tag{26}$$

If the expression for $E_0(\kappa)$ from Eq. (13), assumed to extend up to scales $\kappa \leq \kappa_d \sim \epsilon^{1/4}/\nu^{3/4}$, is substituted in the definitions of S_0 and G_0, we find

$$\begin{aligned} S_0 &= -\int_0^\infty 4\nu\kappa\mathcal{F} \sim -\nu\kappa_d^2 \sim -\kappa_d^{2/3} \\ G_0 &\sim \nu^2\kappa_d^{10/3} \sim \kappa_d^{2/3} \end{aligned} \tag{27}$$

indicating the $\kappa_d^{2/3} \sim Re^{1/2}$ divergence of vortex stretching S_0 and enstrophy destruction G_0 identified by Tennekes and Lumley (1972).

The steady state balance obtained by setting $\dot{\epsilon} = 0$ and, since $\Pi(\kappa,t)$ is concentrated near large scales, $\Pi = 0$ in Eq. (8) is $S + G = 0$. But this balance obviously remains correct for the time-dependent expression Eq. (13), therefore also $S_0 + G_0 = 0$; thus, in the limit of slow spectral variation described by Eq. (13), the leading order $Re^{1/2}$ divergences in the exact ϵ equations cancel.

Simple calculations show that

$$\begin{aligned} S_1 &\sim \nu\kappa_d^{4/3} \sim \kappa_d^0 \\ G_1 &\sim \nu^2\kappa_d^{8/3} \sim \kappa_d^0 \end{aligned} \tag{28}$$

therefore,

$$\dot{\epsilon} \approx -S_1 - G_1 \sim \kappa_d^0 \tag{29}$$

which demonstrates that, again in the limit of slow spectral variation described by Eq. (13), there is in fact an $O(Re^0)$ difference between $-S$ and G. A special case of this cancellation was noted by (Rubinstein *et al*, 2004).

We note that the derivation of an ϵ equation crucially depends on the existence of Yoshizawa's correction term: if the spectrum were exactly a local Kolmogorov spectrum as in Eq. (13), the balance $S + G = 0$ would hold identically, and the ϵ equation would reduce to $\epsilon = P$. However, Yoshizawa's correction was found from the solution of a linear integral equation; it does not correspond to any obvious correlation. This makes single-point modeling of the exact ϵ equation problematic.

THE GENERAL COMPATIBILITY EQUATION

The special structure of $\mathcal{L}^\dagger$ makes it possible to replace Eq. (21) by a linear second order ordinary differential equation. Define

$$\begin{aligned} A &= \int_\kappa^\infty \sqrt{\frac{E_0(p)}{p^3}}dp & B &= \int_0^\kappa (\kappa')^2 E_0(\kappa')d\kappa' \\ C &= \sqrt{\kappa^3 E_0(\kappa)} & F &= \int_\kappa^\infty \sqrt{\frac{E_0(p)}{p^3}}\Psi(p)dp \\ G &= \int_0^\kappa \kappa^2 E_0(\kappa)\Psi(\kappa)d\kappa \end{aligned} \tag{30}$$

Note that

$$F' = A'\Psi \qquad G' = B'\Psi \tag{31}$$

Rewrite $\mathcal{L}^\dagger$ using these definitions as

$$\mathcal{L}^\dagger[\Psi(\kappa)] = \kappa^2(A\Psi - F) + \frac{1}{2C}(G - B\Psi) = 0 \tag{32}$$

Eq. (32) states that

$$2C\kappa^2(A\Psi - F) + (G - B\Psi) = 0 \tag{33}$$

One κ derivative gives

$$[2C\kappa^2]'(A\Psi - F) + 2C\kappa^2 A\Psi' - B\Psi' = 0 \tag{34}$$

which can be rearranged as the ordinary differential equation for Ψ,

$$A\Psi' = \left[\frac{B - 2C\kappa^2 A}{(2C\kappa^2)'}\Psi'\right]' \tag{35}$$

It is again evident that $\Psi' = 0$ satisfies Eq. (35); if $\Psi' \neq 0$, then a second solution is found from

$$\Psi' = \frac{c}{Q}\exp\left\{\int \frac{A}{Q}d\kappa\right\} \tag{36}$$

where

$$Q = \frac{B - 2C\kappa^2 A}{[2C\kappa^2]'} \quad (37)$$

It is not possible to conclude that a second consistency condition exists, because the process of elimination that started from Eq. (33) can also begin instead from $(A\Psi - F) + [2C\kappa^2]^{-1}(G - B\Psi) = 0$. However, this path proves to lead to the same result as Eq. (36); thus, we can conclude that there are indeed two consistency conditions for the solution of Eq. (17), the energy equation Eq. (7) and Eq. (20) with Ψ defined as the solution of Eq. (36). The derivation of a satisfactory two equation model will depend on the possibility of expressing this second equation in terms of the single point moments $k(t)$, $\epsilon(t)$, and $P(t)$. Investigation of this possibility is in progress.

SELF-SIMILAR EVOLUTION

The ϵ equation is usually derived phenomenologically. If we agree that $\dot{\epsilon}$ should be a function of k, ϵ, and P, the general form

$$\dot{\epsilon} = \frac{\epsilon}{k}(C_{\epsilon_1} P - C_{\epsilon 2}\epsilon) \quad (38)$$

follows from dimensional analysis. Values of $C_{\epsilon 1}$ and $C_{\epsilon 2}$ are found by requiring the two-equation model Eqs. (7) and (38) to give the correct results for the decay rate of isotropic turbulence and the growth rate of homogeneous shear flow, assumed for this purpose to be universal constants.

It is important to show that this argument does not contradict our previous conclusion. A crucial feature is that this argument relies on calibration to *self-similar* flows. Under conditions of self-similarity, it is trivial that the frequency scales $\dot{\epsilon}/\epsilon$, P/k, and ϵ/k are linearly related; therefore, for a self-similar flow, Eq. (38) is a kinematic necessity, not a general dynamic statement. Indeed, one can go considerably further and state that two-equation models can only rigorously apply to self-similar states (Clark and Zemach, 1998) because only then is the infinity of time scales in a turbulent flow reducible to a finite number.

In (Rubinstein and Clark, 2005), general classes of self-similar states of time-dependent isotropic turbulence are considered. A typical example is the class of self-similar states for which $P(t) = At^a$ and $\epsilon(t) = Bt^a$ at long times. These solutions can be obtained in forced isotropic turbulence by making the production grow as indicated, and forcing at a scale $\lambda_0(t) = Ct^{-a/2-3/2}$. Note that the forcing scale is time-dependent. The relation between the constant C in the forcing scale and the asymptotic ratio of production to dissipation, A/B depends on the model.

The constants in a two-equation model consistent with this state satisfy

$$\frac{a}{a+1} = \frac{C_{\epsilon 1}(A/B) - C_{\epsilon 2}}{(A/B) - 1} \quad (39)$$

This equation can be interpreted in two different ways: given the production growth rate exponent a and the proportionality constant C in the growth law for the forcing scale, this equation determines all of the values of the constants $C_{\epsilon 1}$ and $C_{\epsilon 2}$ consistent with these parameters; alternatively, given the constants in the ϵ equation, it determines values of a and C which define possible self-similar states for the two-equation model defined by Eqs. (7) and (38).

Clearly, any choice of model constants selects a class of consistent self-similar states, but not every self-similar state is consistent with a given choice of constants. A similar phenomenon has been observed in shearless diffusion in (Oberlack and Gunther, 2003). The underlying reason is that the Navier-Stokes equations admit infinitely many scaling groups and corresponding similarity solutions, whereas a two-equation model can only admit a restricted subclass. It should be noted that spectral closures, in particular the Heisenberg model used here, are consistent with all the scaling invariances of the Navier-Stokes equations.

We note that it is not crucial that the length scale of the forcing be prescribed; these self-similar states can also be obtained by requiring the forcing scale to be a fixed multiple of the peak scale of the energy spectrum; in this way, the turbulence 'chooses' its integral scale, which is not fixed in advance (Rubinstein and Clark, 2005).

CONCLUSIONS

It is uncertain whether a universal equation govening the evolution of ϵ and κ_0, or equivalently of k and ϵ, exists apart from the energy balance Eq. (7). This situation contrasts unfavorably with kinetic theory, in which closed equations for the descriptors of a local Maxwellian, namely the hydrodynamic moments, can indeed be derived through the Chapman-Enskog expansion. These equations are of course the Navier-Stokes equations themselves, which could be described from the present viewpoint as a 'five equation model' for the Boltzmann equation.

Several reasons can be identified which explain this distinction between kinetic theory and turbulence theory. In kinetic theory, the scale separation between thermal fluctuations and hydrodynamic motion is so great that the effect of the thermal fluctuations on the hydrodynamic motion admits description through transport coefficients alone. In turbulence, a continuum of relevant scales of motion separates the long time scales on which κ_0 evolves from the small scales on which ϵ evolves. This lack of strong scale separation impedes the derivation of evolution equations for descriptors of slowly varying spectral evolution, but cannot yet be said to rule it out entirely.

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