TWO-POINT CLOSURES AND TURBULENCE MODELING

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ABSTRACT

The main ideas underlying the derivation of the usual two-point closures are reviewed. The Direct Interaction Approximation (DIA) and related closures, and the Eddy-Damped Quasi Normal Markovian (EDQNM) model are briefly presented. Particular emphasis is given to the application of closures to homogeneous anisotropic turbulence. The relations between two-point closures and other types of turbulence models, in particular subgrid models for Large-Eddy Simulations, are briefly presented. Finally, the possible extensions of two-point models to inhomogeneous turbulence are discussed and some representative results are given.

INTRODUCTION

Two-point closures, also called spectral models or analytical theories of turbulence, are statistical approaches. They are based on ensemble averages and the Reynolds decomposition. In this respect, they can be considered to belong to what is nowadays referred to as RANS modeling (for Reynolds Averaged Navier Stokes). One might further categorize them as "sophisticated" RANS models. On the other hand, two-point closures can also claim the status of "theories" in the sense that at least some of them are self-consistent approaches free of *ad hoc* parameters or empirical constants that must be determined by comparing to experiments[1]. They are based on the Navier Stokes equations and an acceptable number of reasonable assumptions. They can be regarded as expansions about Gaussianity of the probability density function of the velocity fluctuation u_i[2](Kraichnan, 1977).

Two-point closures were proposed and applied in the late 50's and in the 60's (1970 for the EDQNM model). As far as isotropic turbulence is concerned, most of the results of closures were obtained in the 60's or 70's (see for example Lesieur and Schertzer, 1978). From the theoretical point of view, the LHDIA (Kraichnan, 1965) appears as the major achievement in the closure strategy and the limitations of closures are well identified : they fail to take internal intermittency into account. It is the author's belief that closures have nevertheless contributed considerably, even recently, to our understanding of anisotropic turbulence and of its interaction with "external" effects such as mean shear or solid body rotation. The oral presentation will try to illustrate this particular aspect of closures. The possible extensions of two-point modelling to inhomogeneous turbulence will also be discussed (see the last section of the present paper).

[1] It is not the claim of the author that this statement should in any sense be regarded as a definition of what could be considered a real theory of turbulence, the existence of which can still be debated

[2] It should be noted that Gaussianity is not mandatory and that one could imagine closures derived as expansions about other types of statistical distributions, although the author is not aware of any such extensions of closures.

BASIC QUANTITIES AND EQUATIONS

One characteristic of two-point closures is that their precise formulation generally requires rather heavy and intricate equations. It is beyond the scope of the present paper to reproduce equations that can nowadays be found readily in textbooks (Leslie, 1973), (Lesieur, 1990), (Mathieu and Scott, 2000).

The basic object in the two-point description of turbulence is the velocity correlation at two points $\vec{x}$ and $\vec{x'}$:

$$< u_i(\vec{x})u_j(\vec{x'}) > \tag{1}$$

in which $<>$ denotes an ensemble average.

In the case of single time closures like EDQNM (Orszag, 1970), the correlation involves only velocity fluctuations at one time t, whereas for two-time theories such as DIA, the correlation is defined at t and t' :

$$< u_i(\vec{x},t)u_j(\vec{x'},t') > \tag{2}$$

In the case of homogeneous turbulence, the correlation is a function of the separation vector $\vec{r} = \frac{\vec{x'}-\vec{x}}{2}$ only :

$$< u_i(\vec{x},t)u_j(\vec{x'},t') >= f_{ij}(\vec{r},t,t') \tag{3}$$

Fourier transformation of the two-point correlation leads to the spectral tensor :

$$\Phi_{ij}(\vec{K},t,t') \tag{4}$$

In the Direct Interaction Approximation (Kraichnan, 1959), a second important quantity is introduced : the response function or propagator $G_{ij}(\vec{K},t,t')$. It is defined as the response δu_i at time t to a perturbation δf_j introduced at t'. The closed set of equations is not reproduced here. It is a set of integro-differential equations for Φ_{ij} and G_{ij}.

It is known that the DIA equations are not invariant under a random Galilean transformation, and that, as a consequence, they do not predict the $K^{-5/3}$ Kolmogorov scaling for the turbulent kinetic energy spectrum in the inertial range. They lead to a spurious $K^{-3/2}$ spectrum instead. It is beyond the scope of the present paper to go into the details of the Lagrangian History version of the Direct Interaction Approximation (LHDIA), but it must be stressed that recasting the theory in a properly Galilean invariant framework led Kraichnan (1965) to propose a theory that has the advantage of predicting both the $K^{-5/3}$ inertial range and an acceptable value of the associated Kolmogorov constant. An alternative procedure was proposed by Kaneda (1981), who introduced an auxiliary Lagrangian position function.

One-time two-point theories are simpler and require less computational effort to be numerically integrated. They are also less complete in the sense that they generally require the specification a spectral time scale, or of a spectral damping term in the case of the Eddy Damped Quasi-Normal Markovian model (EDQNM, Orszag, 1970). There are two different ways to derive the EDQNM model. It can be presented as a simplification of a two-time DIA formalism in which one simply assumes the two-time correlation to be an exponentially decreasing function of the time separation with a damping time scale compatible with the existence of a Kolmogorov inertial range (see equation (11) below). It can also be presented directly; by working with one-time quantities only. In this case, the damping coefficient is introduced as the inverse of a relaxation time for the triple velocity correlation (Orszag 1970).

The equations of the EDQNM model are here recalled in the case of isotropic turbulence where their formulation remains simple. Isotropy leads to a description in terms of the turbulent kinetic energy spectrum as a function of wavenumber alone:

$$E(K) = \frac{1}{2} \int \Phi_{ii}(\vec{K}) d\Sigma \tag{5}$$

in which Σ denotes integration over a spherical shell of radius K. $E(K)$ is the well known turbulent kinetic energy spectrum, which is governed by the Lin equation :

$$\frac{\partial E(K,t)}{\partial t} = -2\,\nu\, K^2\, E(K,t) + T(K,t) \tag{6}$$

in which T is the non linear transfer term whose expression depends on the closure. In the case of the EDQNM model, T is given by

$$T(K,t) = \int_{\Delta(k)} \theta_{KPQ} \frac{xy + z^3}{Q} E(Q,t)\{K^2 E(P,t) - P^2 E(K,t)\}\; dPdQ \tag{7}$$

in which the integral is over all the triads such as $\vec{K}$, $\vec{P}$ and $\vec{Q}$ can form a triangle. x, y and z are respectively the cosines of the angles opposite to $\vec{K}$, $\vec{P}$ and $\vec{Q}$ in this triangle. θ_{KPQ} is the triadic time scale defined as :

$$\theta_{KPQ} = \frac{1}{\nu(K^2+P^2+Q^2) + \eta(K) + \eta(P) + \eta(Q)} \tag{8}$$

The quantity $\eta(K)$ is the damping coefficient. As stated above, it is introduced as a damping of the triple correlation in the direct derivation of EDQNM, whereas when deducing EDQNM from DIA, it appears as the inverse of the time scale associated with the two-time correlation ;

$$\Phi_{ij}(\vec{K},t,t') = \Phi_{ij}(\vec{K},t,t)\; exp(-\eta(K)(t-t')) \tag{9}$$

Compatibility of the model with a Kolmogorov inertial range :

$$E(K) \propto \varepsilon^{2/3} K^{-5/3} \tag{10}$$

requires the scaling:

$$\eta(K) \propto \varepsilon^{1/3} K^{2/3} \tag{11}$$

Different formulations of η compatible with (11) in the inertial range can be used, the prefactor (the only adjustable constant in the model) being expressed as a function of the Kolmogorov constant (by assuming that the energy flux in the cascade must be equal to the turbulent dissipation). Other two-point one-time closures exist. Among them, the Test Field Model (TFM) has to be mentioned (Kraichnan, 1971). The TFM leads to the same expression for the transfer term $T(K)$ than EDQNM (equation (7)). The two models only differ in the specification of the coefficient $\eta(K)$ which is less heuristic for the TFM.

Figure 1 shows an example of time evolution of the turbulent kinetic energy spectrum in the case of homogeneous isotropic turbulence. In this case the EDQNM closure is used. It can be observed that a $K^{-5/3}$ inertial range develops in the spectrum.

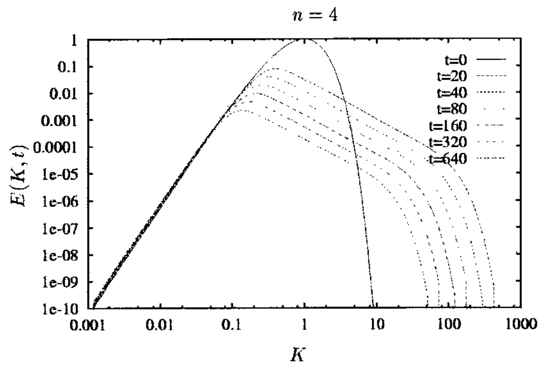

Figure 1: Turbulent kinetic energy spectrum for isotropic turbulence. EDQNM model. Results plotted at different times.

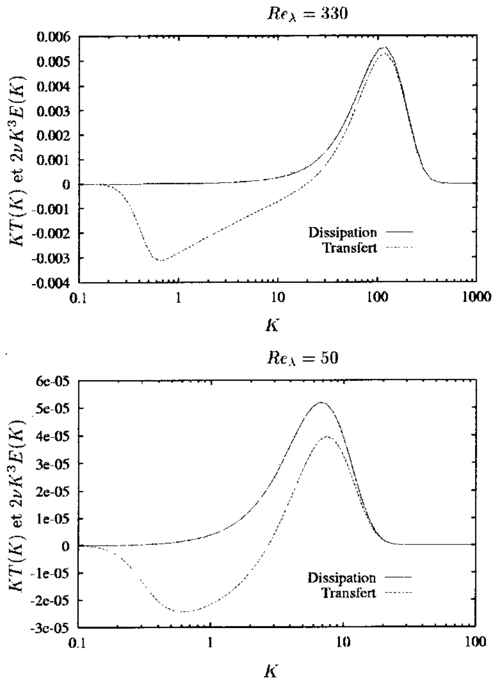

Figure 2: Energy transfer and dissipation spectra at moderate (top) and small (bottom) Reynolds number (isotropic turbulence; EDQNM model).

Figure 2 shows the shape of the non linear transfer term corresponding to expression (7), in the case of decaying isotropic turbulence. Also plotted in the figure is the dissipation spectrum $2\nu K^2 E(K)$. In the upper part of the figure, it can be observed that at sufficiently high Reynolds number ($Re_\lambda = 330$) there is a balance between transfer and dissipation at large wave-numbers. For a smaller Reynolds number ($Re_\lambda = 50$), it can be observed that the transfer remains smaller than the dissipative contribution. This Reynlods number effect can be characterized by evaluating the ratio between the energy flux ε_f in the cascade and the dissipation ϵ. This ratio is plotted as a function of Re_λ in Figure 3.

CLOSURES FOR HOMOGENEOUS ANISOTROPIC TURBULENCE

Homogeneous anisotropic turbulence is simple enough to be treated using two-point closures. It is however much richer than simple isotropic turbulence since interesting practical effects, like turbulence production by mean shear, can be studied. Homogeneous anisotropic turbulence might well be the situation in which two-point closures have revealed the most useful results. It must be pointed out that Rapid Distortion Theory (RDT) can be considered as the simplest two-point closure, in which the unknown non linear terms are modeled simply by replacing them by zero. Indeed, in the particular case of isotropic turbulence, RDT leads to a trivial model in which the turbulent kinetic energy will simply remain constant and turbulence will never decay in time (or will decay at an unrealistic viscous rate). Once one considers turbulence in the presence of a mean velocity gradient, the situation is drastically different and RDT is known to lead to valuable insights on the interaction of turbulence with the mean flow. It also provides interesting information on terms that must be modeled at the level of a single point description, like the rapid part of the pressure strain correlation.

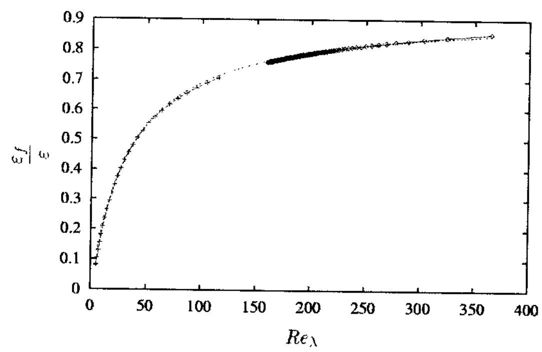

Figure 3: Ratio of the energy flux to the turbulent dissipation as a function of the Reynolds number.

More realistic closures, such as EDQNM, combine the advantage of exactly accounting for the rapid terms, as RDT does, with taking into account the non linear transfer of energy to the small scales (and the associated slow part of the pressure strain term) in a modeled but quite realistic way. Since the energy transfer is present in the theory, there is no need to introduce an arbitrary ε equation as in one-point models. The advantage of two-point closures is clear : once the triple correlations are modeled, no other assumptions have to be made. One could say that the closure problem is then well formulated, in the sense that the only assumptions required pertain to the triple correlations.

The first attempts to extend the EDQNM two-point closure to turbulence in the presence of a mean shear were made by Cambon et al (1981) and Bertoglio (1981). Later, the EDQNM closure was revealed to be very helpful in the prediction and understanding of homogeneous turbulence subjected to solid body rotation (Cambon and Jacquin, 1989) and stable stratification (Staquet and Godeferd, 1998). In both cases, the dynamics of the flow is characterized by the interaction between waves and the turbulent motion. Another situation in which waves are present that was studied using EDQNM (Bertoglio et al., 2001) (or a modified version of EDQNM, Fauchet et al., 1999) is the case of weakly compressible turbulence. It has to be pointed out that when waves are present, the EDQNM model shows strong similarities with weak turbulence theories (see Zakharov et al, 1992).

REYNOLDS NUMBER EFFECTS : CLOSURES AND SCALINGS

When closures were first proposed and applied, in the 50's and 60's, there was no alternative to predict the details of the behavior of turbulence, and in particular of the turbulent spectrum. Obviously the situation has drastically changed, and today most of the situations to which closures can be applied (homogeneous turbulence for example) can very easily be treated with Direct Numerical Simulation. Coding the equations requires less effort for the Navier Stokes equation than for the integro-differential set of equations resulting from theories like DIA, and the results can also be considered free of any uncertainty since no approximations are made. The interest of closures nowadays is related to the fact that they can be applied to high Reynolds number turbulence which is not the case for DNS. The strategy is then to apply a closure to a given situation, validate the results of the closure at small or moderate Reynolds numbers by comparisons with DNS, and then use the closure to explore the high Reynolds number domain.

One has also to mention that closure can be helpful in suggesting the relevant parameters on which to build theoretical scaling. Often, the analytical formulations of closures naturally provide a route to identifying the relevant parameters.

An example of comparison of closure with DNS is given in Figure 4 in which the equivalent of the $C\varepsilon_2$ parameter in the ε equation. It is seen that the agreement with DNS is satisfactory at small Re_λ and that the closure accounts for the Reynolds number effects.

Figure 5 illustrates another case of comparisons between closures and DNS (and in this case LES results are also plotted). In this case, the decay of isotropic turbulence in a bounded domain is investigated (Touil et al. 2002). The turbulent kinetic energy first decay with a classical ($n = -10/7$) exponent, until the integral length scale reaches the size of the domain (the length scale saturation in the Helium experiment of Skrbek and Stalp (2000)). Later, the decay exponent is found to be -2. The symbols in the plot are DNS and LES data, and the solid line represents the EDQNM computation. It appears that the observed behaviors are in agreement. It also appears that the closure permits to explore a much larger range of parameters than the DNS (and even than LES).

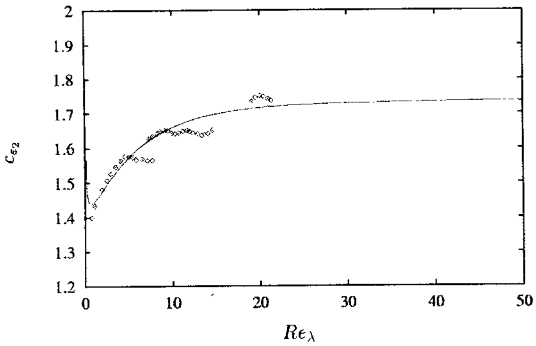

Figure 4: Variation of $C_{\epsilon 2}$ with the Reynolds number. EDQNM predictions compared to DNS data.

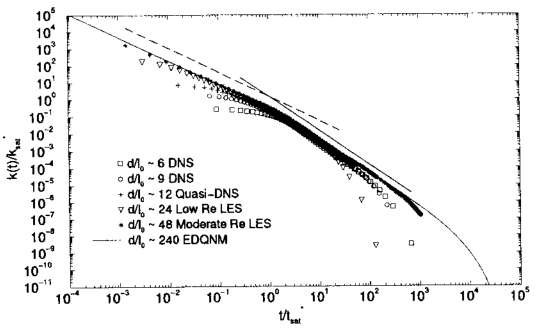

Figure 5: Decay of turbulence in a bounded domain. Example of complementarity between DNS LES and a closure (EDQNM).

CLOSURES AND SUBGRID MODELS FOR LES

Since two-point closures reproduce the cascade of energy to the small scale dissipative range of the spectrum that is missing in Large-Eddy Simulations, they were among the first tools used to model the subgrid terms. Kraichnan (1976) was the first to derive a subgrid model based on a two-point closure and to propose an expression for a subgrid eddy viscosity. Further investigations of the subgrid problem based on two-point closures were performed by Leslie and Quarini (1979). The most popular subgrid model deduced from a closure is certainly that of Chollet and Lesieur (1981), in which the subgrid viscosity is expressed as a function of the energy spectrum at the filter cut-off wave-number K_C :

$$\nu_t^0 = C(\frac{E(K_C)}{K_C})^{1/2} \tag{12}$$

in which C is a constant ($C = 0.267$). The EDQNM closure shows that ν_t is indeed a function of the wave-number :

$$\nu_t = \nu_t^0 f(K/K_C) \tag{13}$$

but that f is approximatively constant and equal to unity over a large range of wave-numbers ("plateau"). Only in the vicinity of the cut-off does f increase ("cusp" effect).

When applied to subgrid modelling, two-point theories also suggest the presence of "backscatter" from small scales to large scales. Using a stochastic model for EDQNM, Bertoglio (1984) proposed modeling the subgrid backscatter by adding a random force term to the classical subgrid viscosity term (see also Chasnov, 1991). A formulation in physical space for the subgrid backscatter was later derived by Leith (1990).

TWO-POINT MODELING AND INHOMOGENEOUS TURBULENCE

There is no theoretical difficulty in extending two-point closures to non homogeneous turbulence. Indeed, the two-point closure formulations were written for inhomogeneous fields several decades ago (see for example Kraichnan, 1972 in the case of the Test Field Model), but due to the complexity of the resulting set of equations they could hardly be used directly or numerically integrated. Only a few attempts have to be mentioned (see for example Dannevik 1984, in the particular case of axisymmetric turbulence with only one direction of inhomogeneity). The major difficulty originates from the fact that relation (3) is no longer valid for inhomogeneous turbulence and that the two-point correlation, instead of being a function of the separation vector only, remains a function of two vectors $\vec{r}$ and a position vector, for example the midpoint position vector $\vec{X} = \frac{\vec{x}+\vec{x}'}{2}$. After Fourier transformation with respect to $\vec{r}$, the spectral tensor is then

$$\Phi_{ij} = \Phi_{ij}(\vec{K}, \vec{X}) \tag{14}$$

showing that one has to work with a tensor in a space whose dimension is 6 (7 for a time evolving problem and even 8 for a two-time description of a time evolving problem).

When a weak inhomogeneity assumption is introduced, the equations can be simplified. This is the route followed by several authors (Burden 1991, Laporta 1995). An alternative is to introduce a two scale expansion (Yoshizawa 1990) (Rubinstein and Erlebacher, 1997). This type of approach usually introduces very few new adjustable parameters, but generally can be applied only to rather simple situations. For example, in the model of Laporta (1995), only the classical EDQNM constant for isotropic turbulence is involved and the model could reproduced the time evolution of a shearless turbulence mixing layer (corresponding to the experiment of Veeravalli and Warhaft 1989). Another example of the application of the model of Laporta is described in Parpais et al. (1995), where the model is applied to diffusive turbulence generated by oscillating a grid in a tank (see the sketch in Figure 6). The model leads to a power law decay of the turbulent kinetic energy as a function of the distance from the grid with an exponent close to -3, as appears in Figure 7, in agreement with DNS data.

When considering applications to more complex geometries, one can hardly imagine that they could be treated without having to introduce more drastic simplifying assumptions. This is the route followed by Besnard et al (1990) or Parpais (1997) and Touil (2002). In both the model of Besnard et al. and the model of Touil, the basic quantity is the spectral tensor integrated over the directions of the wave-vector :

$$\varphi_{ij}(K, \vec{X}) = \int \Phi_{ij}(\vec{K}, \vec{X}) d\Sigma \tag{15}$$

in which Σ is the spherical shell of radius K (see also Cambon et al. 1981). There is a price to pay for this simplification : the pressure velocity correlation term becomes an unknown term requiring modeling, as in the classical one point approach.

The transport of turbulent kinetic energy by triple correlations in this type of simplified approach is usually treated heuristically by a diffusive modeled form. After all of these simplifications, the resulting models can hardly be called two-point closures. For example, the model of Touil (2002) shares many assumptions with the Launder Reece and Rodi

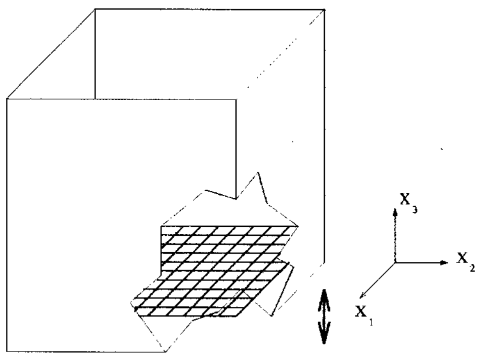

Figure 6: Scheme of a diffusive turbulence experiment - turbulence generated by oscillating a grid in a tank.

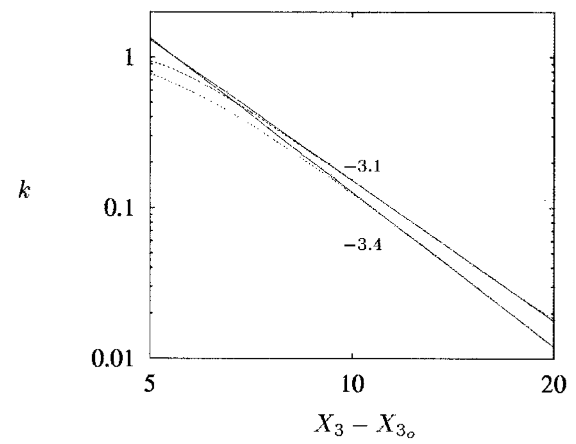

Figure 7: Spatial decay of the turbulent kinetic energy as a function of the distance from the grid. Results of the EDQNM model of Laporta (1995) compared with decay power laws.

(1975) type of one point model. However, it must be pointed out that for isotropic turbulence, the model is identical to the EDQNM closure. An example of the application of the model of Touil is given in Figures 6 and 7. In this case the flow around an airfoil at an angle of attack of 13 degrees is computed. At each point of the physical domain, the transport equation for φ_{ij} is solved. In Figure 7 the energy spectra are plotted along a mean streamline that passes near the airfoil and goes downstream in the wake. In Figure 6, the ratio between the spectral energy flux ε_f and the turbulent dissipation ε is plotted. It must be stressed that when applied to inhomogeneous turbulence, the two-point correlation approach could also be used without Fourier transformation, directly in the physical space. This is the route followed by Oberlack (1997).

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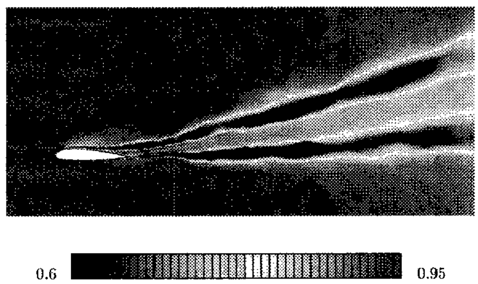

Figure 8: Flow around an airfoil at an angle of attack, computed with the model of Touil; visualization of the ratio of the energy flux to the turbulent dissipation

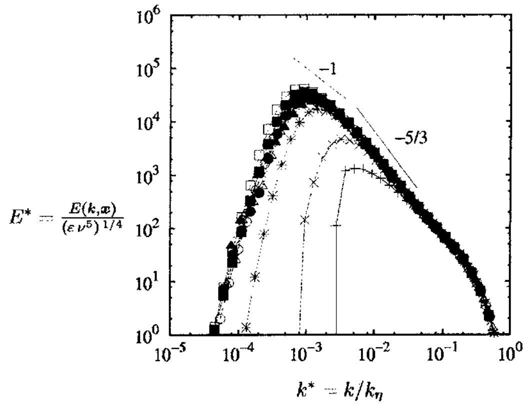

Figure 9: Flow around an airfoil; turbulent kinetic energy spectra along a mean streamline.

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