ON THE DEVELOPMENT OF SPATIOTEMPORAL STRUCTURE 
IN A TWO-DIMENSIONAL TURBULENT JET

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INTRODUCTION

In recent years, the generation and collapse process of the coherent structure in a turbulent flow have been observed in many experiments and numerical simulations (see Holmes et al. (1996), Silva and Mettas (2002a) for example). Since these coherent structures cause the oscillation, the frictional drag and the sound noise, etc., the understanding of these physical characteristics is very important for industrial applications. In this study, we adopted a plane jet which is fundamental and easy to handle as a research subject of the coherent structure. As the experimental technique to extract the coherent structures, the Karhunen-Loève expansion (hereafter, called the KL expansion) which is also referred to as POD (Proper Orthogonal Decomposition) is an useful mathematical tool. It is known that the KL expansion can extract the spatial eigenfunctions (i.e., 'modes') which represent the coherent structures if they have a dominant percentage of the turbulent kinetic energy (Holmes et al., 1996). In our last paper (Sakai et al., 2002b), we reported the downstream change of distributions of the eigenvalues and the eigenfunctions in the cross-streamwise direction only by the KL expansion. In this paper, mainly by the combination of the KL expansion and the Fourier transform, the structure development of a plane jet has been investigated from a viewpoint of frequency space besides the physical space. Further, the shear stress spectrum and spatial correlation of velocity are examined and the jet structure is discussed from a viewpoint of these joint statistics.

SUMMARY OF THE COMBINED ANALYSIS OF THE KL EXPANSION AND THE FOURIER TRANSFORM

By the combination of the KL expansion and the Fourier transform, the orthogonal decomposition for both space and frequency can be performed (George (1990), Delville et al. (1991)). In this study, the KL expansion was made to the cross-streamwise direction (x2-direction in figure 2) and the Fourier transform was applied to the time space.

The problem to solve may be written as follows:

\[ \int I R(x_2, x_2', \tau) \phi(n)(x_2', \tau)dx_2' = \lambda(n)(\tau) \phi(n)(x_2, \tau), \]  
\[ R(x_2, x_2', \tau) = \langle u(x_2, t)u(x_2', t') \rangle, \]

where \( \tau = t - t' \) is the time difference, \( R(x_2, x_2', \tau) \) is the two-point correlation of the velocity component, i.e., \( u \) (the \( x_1 \)-component, see figure 2), \( \phi(n)(x_2, \tau) \) is the eigenfunction in the \( x_2, \tau \) space and \( \lambda(n)(\tau) \) is the eigenvalue. Now we describe the time Fourier transform of \( R, \phi(n) \) and \( \lambda(n) \) as \( \Phi, \phi(n) \) and \( \lambda(n) \), respectively. For instance, \( R(x_2, x_2', \tau) \) is transformed as follows,

\[ \Phi(x_2, x_2', f) = \int R(x_2, x_2', \tau)e^{-i2\pi f \tau} d\tau, \]  
\[ \Phi(x_2, x_2', f) = \int \Phi(x_2, x_2', f) \phi(n)(x_2', f) dx_2' = \lambda(n)(f) \phi(n)(x_2, f). \]

Since in the present experiments the velocity field was measured by the hot-wire probes, the kernel in equation (4) can be known only at discrete positions where probes are set up. Then the equation (4) can be replaced by the discrete eigenvalue problem of the matrix \( \Phi \) which has the eigenvalue \( \lambda(n) \) and the eigenfunctions \( \phi(n) \). Here we must solve the complex eigenvalue problem because each element of the matrix \( \Phi \) is a complex number. But since \( \Phi \) is the Hermitian matrix, the eigenvalues are real numbers and the eigenfunctions become complex functions. The KL expansion provides an important property that the turbulent kinetic energy can be expressed as the sum of the eigenvalues, i.e.,

\[ E = \int (u^2) dx = \sum_{n} \lambda(n)(f). \]

Therefore the eigenfunction with the maximum eigenvalue contains the largest amount of kinetic energy. Usually, according to the magnitude of the eigenvalue, the eigenvalue and eigenfunction are ordered such that \( (\lambda^{(1)}, \phi^{(1)}), (\lambda^{(2)}, \phi^{(2)}), \ldots, (\lambda^{(n)}, \phi^{(n)}) \). Using eigenvalues \( \lambda(n) \) and eigenfunctions \( \phi(n) \) obtained by this analysis, the power spectrum \( S \) is reconstructed from the following equation

\[ S(x_2, f) = \sum_{n=1}^{\infty} \lambda(n)(f) \phi(n)(x_2, f)^2. \]

EXPERIMENTAL APPARATUS AND CONDITIONS

Figure 1 shows the sketch of the experimental apparatus. The jet was issued from the skimmer which has the height of \( d = 12\text{mm} \) and the aspect ratio of 19.7 (see figure 2). The Reynolds number \( Re(=U_0d/\nu) \) is adjusted to 16,000 where the exit velocity \( U_0 \) is about 20m/s. For the velocity measurements, the X-type hot wire probe was used. The tungsten wire of 5μm was attached with the thin tip of the piano wire of a diameter of 0.5mm by means of the spot welding. Figure 2 shows the coordinate system. The streamwise (downstream) coordinate is \( x_1 \), the vertical
(cross-streamwise) coordinate is \( z_2 \) and the spanwise coordinate is \( z_3 \). Firstly, the one-point measurements have been made along the jet centerline and in the cross-streamwise direction at several streamwise locations, i.e., at \( x_1/d = 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 8.0, 10.0, 15.0, 20.0, 25.0, 30.0, 35.0, 40.0 \). Next the two-point measurements have been performed, and the data examined by calculating the spatial correlation or making the combined analysis of the KL expansion and the Fourier transform. The locations of the two-point measurements are \( x_1/d = 2.0, 3.0 \) (in the potential core region), 4.5 (in the beginning of the interaction region), 6.0 (in the end of the interaction region), 10.0 and 20.0 (in the self-preserving region). The 21 measurement points are set up in one section and the interval of each measurement points is adjusted to 2.5b/10 (b: half-width of the radial profile of the mean streamwise velocity). The cross-spectrum is calculated from the discrete finite samples by means of FFT. Because of the sampling frequency 10kHz, the frequency domain of the cross-spectrum is limited to 5kHz of the Nyquist frequency. The frequency resolution \( \Delta f \) is about 2.4Hz which is obtained by dividing the Nyquist frequency into 2048(=2^11) parts. Consequently, the eigenfunctions \( \lambda_{nf}(f) \) of 2047 except for \( f = 0.0Hz \) are calculated from the cross-spectrum matrix of 21\( \times \)21.

FUNDAMENTAL CHARACTERISTICS OF A PLANE JET IN PRESENT EXPERIMENTS

The first moment (the mean streamwise velocity \( U \) and the mean cross-streamwise velocity \( V \)) and the second moment (the streamwise r.m.s. velocity \( u' \) and the cross-streamwise r.m.s. velocity \( v' \)) of the jet velocity field are investigated. Although these figures are not shown here, we ascertained that the centerline streamwise mean velocity \( U_m \) for \( x_1/d > 6.0 \) is approximated by

\[
\left( \frac{U_m}{U_0} \right)^{-2} = 0.13 \left( \frac{x_1}{d} + 3.4 \right).
\]

Then the half-width \( b \) for \( x_1/d > 6.0 \) is approximated by

\[
\left( \frac{b}{d} \right) = 0.11 \left( \frac{x_1}{d} + 2.6 \right).
\]

The cross-streamwise profiles of the first moment and the second moment show the similarity for \( x_1/d > 6.0 \) and \( x_1/d > 10.0 \), respectively. Figure 3 shows the profile of Reynolds stress \( \overline{uu} \). The profile becomes nearly similar for \( x_1/d > 10.0 \) and the peak value exists at \( x_2/b \sim \pm 0.65 \).

![Figure 3: Reynolds stress profiles](image)

ANALYSIS OF THE JOINT STATISTICS

Shear-stress cospectrum

The cross-spectrum \( S_{uv}(f) \) for the Reynolds shear-stress \( \overline{uu} \) is generally a complex function

\[
S_{uv}(f) = K_{uv}(f) - iQ_{uv}(f),
\]

where \( K_{uv}(f) \) is a real part and \( Q_{uv}(f) \) is an imaginary part, and then \( K_{uv}(f) \) and \( Q_{uv}(f) \) are respectively called a cospectrum and a quadrature spectrum. Figure 4 shows the downstream variation of the shear-stress cospectrum near the position of the peak of Reynolds stress and near the centerline at several streamwise locations. The ordinate is normalized by the \( u' v' \) and the integral time scale \( T_{uu} \) of \( u \). The cospectrum near the position of the peak of the Reynolds stress exhibits apparently the -7/3 power range at \( x_1/d = 6.0 \), but the -7/3 power law becomes obscure at \( x_1/d = 20.0 \) and 40.0. On the other hand, the cospectrum near the centerline shows almost the similar distribution at each downstream location and does not exhibit the -7/3 power law. This reason is that the -7/3 power range appears only when both the energy-dissipation rate and the mean-shear rate \( \partial U/\partial x_2 \) have played important roles in characteristics of the turbulent field. The influence of the local mean-shear will become small in the larger downstream.

![Figure 2: Coordinate system](image)
region and the mean-shear has not any explicit effect on the
cospectrum at $x_1/d \approx 40.0$. It is a future subject to investi-
gate the relationship between the $7/3$ power law and the
structure of a plane jet.

(a) at the position near the Reynolds stress’s peak

(b) at the position near the jet centerline

Figure 4: Downstream variation of the cospectrum $K_{uv}(f)$

Spatial correlation of the velocity

Figure 5 shows the spatial correlation coefficient of the
velocity in three regions. The spatial correlation coefficient
is defined by the following equation

$$R_{uu}(r) = \frac{u_i(x_2,t)u_i(x_2',t)}{u_i^2(x_2,t)u_i^2(x'_2,t)},$$

(10)

where $r = x'_2 - x_2$ is the probe interval, and $i$ is 1 or 2
(the summation convection is not used here). We also $u$
and $v$ instead of $u_1$ and $u_2$. The contour lines in this fig-
ure are drawn for every 0.1 and show 1.0 in the probe's
position, where the abscissa and ordinate take the same
value. It is found that the correlation patterns are differ-
ent in each region. Since the pattern in the interaction
region (figure 5(b)) has both features of the potential core
region and the self-preserving region, it turns out well that
this location is in the transition region. On opposite sides
of the jet centerline, the $u$-correlation $R_{uu}$ becomes neg-
ative and the $v$-correlation $R_{vv}$ becomes positive in the
self-preserving region (figure 5(c)). This feature in the
self-preserving region has been also confirmed by other in-
vestigators and this can be attributed to the jet flapping
phenomenon (antisymmetric array of counter-rotating vor-
tices). The experimental results also show that the values
of $R_{uu}$ at $(x_2/b, x'_2/b) = (\pm 0.5, \mp 0.5)$, $(\pm 0.75, \mp 0.75)$
and $(\pm 1.0, \mp 1.0)$ become negative at $x_1/d = 10.0$ and the val-
ues at $(x_2/b, x'_2/b) = (\pm 0.5, \mp 0.5)$ and $(\pm 0.75, \mp 0.75)$ecome negative at $x_1/d = 20.0$. So, we investigated the
downstream variation of spatial correlation $R_{uu}$ and $R_{vv}$
at $(x_2/b, x'_2/b) = (\pm 0.5, \mp 0.5)$ and $(\pm 0.75, \mp 0.75)$ (figure
6). It is found that the spatial correlation in the potential
core region takes the sign opposite to the one in the
self-preserving region. So it is considered that the counter-
rotating vortices are formed symmetrically for the jet cen-
terline in the potential core region. Further, in the distribu-
tion of $u$-spatial correlation in the potential core region (left
side of figure 5(a)), the lobes with the negative peak near
$(x_2/b, x'_2/b) = (\pm 0.25, \pm 1)$ and $(\pm 1, \pm 0.25)$ are observed.
These negative peaks are caused by the same vortex struc-
ture which has the larger speed near the jet centerline and
the smaller speed at an outer edge.

(a) in the potential core region $(x_1/d = 2.0)$

(b) in the interaction region $(x_1/d = 4.5)$

(c) in the self-preserving region $(x_1/d = 20.0)$

Figure 5: Downstream variation of spatial correlation $R_{uu}$,
THE COMBINED ANALYSIS OF THE KL EXPANSION AND FOURIER TRANSFORM

Eigenvalues

Figures 7 and 8 show the downstream variation of eigenvalue distribution. The distribution of the potential core region is different from ones of the other regions. In the potential core region the contributions of the first mode and second mode are very large as compared to ones in other regions, and the sum of first three modes for $u$ and $v$ reaches about 80% and 90%, respectively. Thus first a few modes can make the dominant contribution to the fluctuation energy. This fact suggests that the Galerkin projection of these first a few modes into the Navier-Stokes equations could provide a model that can capture the large-scale dynamics of a plane jet (Aubry (1988), Ukeiley and Glaser (1995)). Namely this analysis method has the possibility for constructing a low dimensional dynamical model of a plane jet. Figures 9 and 10 show the downstream variation of eigenvalues $\lambda_u^{(1)}(f)$ and $\lambda_v^{(1)}(f)$ for the first mode. The ordinates are normalized by the sum of all eigenvalues $\sum \lambda_u^{(1)}(f)$ and $\sum \lambda_v^{(1)}(f)$. So the value of the ordinate shows the contribution of $\sum \lambda_u^{(1)}(f)$ to the fluctuation energy, so that it has the same meaning as the normal power spectrum. In this study, the frequency distribution of the eigenvalue is called the 'eigenvalue spectrum'. From figure 9 and 10, it is noticed that the downstream variations of the eigenvalue spectra are very similar to the downstream variations of the power spectra of the $u$ and $v$ component (see figure 12), i.e., as the location becomes far from nozzle, the contribution of the low frequency range grows and the contribution of the high frequency range decreases. Therefore it turns out that the large-scale (low frequency) structure becomes dominant in the first eigenvalue spectrum as going to the more downstream position. Further, it is also noticed that the -5/3 power law appears like the power spectrum in the self-preserving region.

Eigenfunctions

Figure 11 shows the space-frequency distribution of the first eigenfunction $\phi_u^{(1)}(x_2,f)$ in three regions. The two lateral axes show the frequency and $x_2$ normalized by the half-width $b$, respectively. The vertical axis takes an absolute value of the first eigenfunction because the eigenfunction is a complex function. The first mode contains the largest amount of kinetic energy and the contributions of the energy to $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are respectively 43.7%, 52.3% at
$x_1/d = 2.0$, 28.6%, 32.7% at $x_1/d = 4.5$ and 33.6%, 31.6% at $x_1/d = 20.0$. In the potential core region, the distribution has the peaks at $|z_2/b| = 0.5$ to 0.7 with respect to space, but it has the peaks at various places with respect to the frequency. Its reason seems to be that the vortex structures produced in the potential core region have not interfered mutually yet. In the interaction region, many large peaks appear along the frequency axis in the high frequency range, and it is thought that many small-scale structures are generated. Since the contribution of energy in high frequency range is very small as shown in the first eigenvalue spectra (figures 9 and 10), the structures in this range contain very small energy. However, it is found that the form of overall structure becomes clearer than the one in the potential core region, so it seems that in the interaction region the reconstruction of vortex structures has been progressed and the stable structures have been formed in the self-preserving region. In the self-preserving region, the distribution in the high frequency range show one peak near centerline with respect to space, while in the interaction region the distribution has two peaks. In the self-preserving region, while the distribution for $u$ and $v$ are different in the low frequency range, they show almost the same distribution in the high frequency range. It is supposed that this should be related with the local isotropy in the high frequency range.

**RECONSTRUCTION OF THE POWER SPECTRUM**

Using eigenvalues $\lambda^{(n)}(f)$ and eigenfunctions $\phi^{(n)}(z_2,f)$ obtained by the combined analysis of the KL expansion and Fourier transform, the power-spectrum $S(z_2,f)$ can be reconstructed from equation (6). Figure 12 shows the reconstructed spectrum on $z_2/b = +0.75$ (at the position near the peak of the Reynolds stress) in three regions. In this figure, 'sum of 1st mode' denotes the spectrum which is reconstructed only by the first mode, and 'sum of 3rd modes' denotes the spectrum which is reconstructed by the first three modes. It is found that the reconstruction is efficiently performed in the range that the original spectrum takes a big value. This shows very well the property of the KL expansion by which the flow structure is reconstructed in the order of the larger energy. In the potential core region, since the contributions of the energy for $<u^2>$ and $<v^2>$ are respectively 82.5% and 90.0%, the profile of 'sum of 3rd modes' reproduces mostly the profile of original spectrum. It turns out that this analysis method is more efficient to extract the flow structure with a peak in the spectrum like in the potential core region, more efficiently than the one with the smooth spectrum like in the self-preserving region. In the interaction region and the self-preserving region, both profiles of 'sum of 1st mode' and 'sum of 3rd modes' show the fluctuation in high frequency range. The reason is that the reconstruction is hardly performed in high frequency range because the contribution from the sum of the first three modes reaches about 60~70% and the energy is mainly used for reconstruction of the structure with the large spectrum value. Therefore in the self-preserving region of a plane jet, it is expected that the fluctuation energy of the small-scale structures contain about 30~40%.

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![Figure 11: Downstream variation of the space-frequency distribution of the first eigenfunction](image1)

![Figure 12: Downstream variation of the reconstructed spectrum ($z_2/b = +0.75$)](image2)
CONCLUSIONS

(1) In the interaction region, the shear-stress cospectrum near the position of the peak of the Reynolds stress exhibits the -7/3 power law, but the one near the centerline does not exhibit the -7/3 power law. The influence of the local mean-shear becomes small in the far downstream region, so the cospectrum near the position of the peak of the Reynolds stress at \( x_1/d \sim 40.0 \) does not exhibit the -7/3 power law.

(2) The distributions of the spatial correlation of the velocity are different in three regions. On opposite sides of the jet centerline, in the potential core region the \( u \)-correlation \( R_{uu} \) becomes positive and the \( v \)-correlation \( R_{vv} \) becomes negative, and in the self-preserving region the \( R_{uu} \) becomes negative and the \( R_{vv} \) becomes positive. These results mean that the counter-rotating vortices are formed symmetrically for the jet centerline in the potential core region and these vortex structures are redistributed in the form of the antisymmetric array in the self-preserving region.

(3) From the combined analysis of the KL expansion and Fourier transform, the following conclusions are obtained.

(a) From the downstream variation of the first eigenvalue spectrum, it is found that the contribution of the low frequency range grows and the one of the high frequency range decreases as the downstream location becomes far from nozzle. Therefore it turns out that the large-scale (low frequency) structure becomes dominant in the first eigenvalue spectrum as going to the more downstream position.

(b) In the self-preserving region, the space-frequency distributions of the first eigenfunctions of \( u \) and \( v \) are different in the low frequency range. However they show almost the same distribution in the high frequency range.

(c) From the reconstruction of the power spectrum, it is found that the reconstruction of the structure is performed in the order of the larger fluctuation energy (the large value of the original’s spectrum). In the self-preserving region, the reconstruction is hardly performed in the high frequency range because the contribution from the sum of the first three modes is about 60–70%. Therefore in the self-preserving region of a plane jet, the fluctuation energy of about 30–40% is contained in the small-scale structures.

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