A NEW K_θ-$K_\theta L_\theta$ TURBULENCE MODEL FOR HEAT FLUX PREDICTIONS

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ABSTRACT

A new $k_\theta - k_\theta L_\theta$ two-equation turbulence model for heat flux predictions is described. This model is associated to a new $k - kL$ two-equation turbulence model for Reynolds stress predictions, which forms a four-equation turbulence model. The method used to analyse existing four-equation models or to obtain a new model is presented. The model is expressed in terms of a generic $k - \phi$ / $k_\theta - \phi_\theta$ model where $\phi = k^a \varepsilon^b$ and $\phi_\theta = k_\theta{}^p \varepsilon_\theta{}^q$ can represent any type of turbulence scales. The transport equations include all diffusion and cross-diffusion terms. Relations on the model constants are derived to force the model to respect some basic physical features. It will be shown why existing models fail to reproduce the characteristics of heated APG boundary layer flows and how to build a new model that fulfils all requirements. Some applications of the new $k - kL$ / $k_\theta - k_\theta L_\theta$ model are presented.

INTRODUCTION

Turbulent heat transfer can be characterised by the turbulent Prandtl number defined as $Pr_t = \nu_t/\alpha_t$ where ν_t is the eddy viscosity and α_t is the thermal diffusivity. In most of Reynolds-averaged Navier-Stokes (RANS) codes used in the industry the thermal diffusivity is simply deduced from the eddy viscosity under the assumption that the turbulent Prandtl number is constant. However it can be seen experimentally that this parameter varies inside a given flow and between two different flows: for example it is close to 0.9 in the logarithmic region of a boundary layer (Orlando et al., 1974), to 0.7 in wakes (Antonia and Browne, 1986) and to 0.5 in mixing layers (Chambers et al., 1985).

An easy way to take into account the variation of the turbulent Prandtl number is to transport two thermal turbulent scales which provide the thermal diffusivity, in addition to the two dynamic turbulent scales that provide the eddy viscosity. These models are known as four-equation turbulence models and can be quite easily implemented in RANS codes. Most of existing four-equation turbulence models are $k - \varepsilon$ / $k_\theta - \varepsilon_\theta$ models where k is the turbulent kinetic energy, ε its dissipation rate, $k_\theta = \overline{\theta^2}/2$ is half the temperature fluctuation variance and ε_θ its dissipation rate. It is already well-known that standard $k - \varepsilon$ models are unable to give accurate predictions of the velocity field for adverse pressure gradient (APG) flows (Rodi and Scheuerer, 1986). It will be shown that $k_\theta - \varepsilon_\theta$ models present the same deficiencies from the thermal point of view and do not predict correctly the temperature profile in heated APG boundary layer flows. Moreover these models often predict too sharp profiles at the outer edge of the turbulent flow, contrary to experiments where the velocity and the temperature profiles approach the edge very smoothly.

The theoretical behaviour of four-equation models can be analysed in homogeneous flows and in the different regions of a ZPG or APG boundary layer following the work of Catris and Aupoix (2000) for the dynamic part of turbulence models. This paper presents the extension of this work for the thermal turbulent scales. This approach will be used to analyse existing $k - \varepsilon$ / $k_\theta - \varepsilon_\theta$ models and to develop a fully new model which will be tested and compared to other models in simple flow situations.

GENERIC MODEL

The idea is to deal with a generic model of the form $k - \phi$ / $k_\theta - \phi_\theta$ where ϕ and ϕ_θ are expressed as combinations of k, ε, k_θ and ε_θ as $\phi = k^a\varepsilon^b$ and $\phi_\theta = k_\theta{}^p \varepsilon_\theta{}^q$. ϕ and ϕ_θ can thus represent any type of turbulence scales depending on the choice of the exponents a, b, p and q. Transport equations for the turbulent scales are very general and include all diffusion and cross terms that allow the complete equivalence between any chosen transported scale. All developments presented here will be made in the case of an incompressible high Reynolds number flow.

The transport equations for a generic $k-\phi$ model (Catris and Aupoix, 2000) are:

$$\frac{Dk}{Dt} = P - \varepsilon + \text{div}\left[\frac{\nu_t}{\sigma_k}\underline{\text{grad}k} + \frac{\nu_t}{\sigma_{k\phi}}\frac{k}{\phi}\underline{\text{grad}\phi}\right] \quad (1)$$

$$\begin{aligned}\frac{D\phi}{Dt} &= \frac{\phi}{k}\left(C_{\phi_1}P - C_{\phi_2}\varepsilon\right) \\ &+ \text{div}\left[\frac{\nu_t}{\sigma_\phi}\underline{\text{grad}\phi} + \frac{\nu_t}{\sigma_{\phi k}}\frac{\phi}{k}\underline{\text{grad}k}\right] \\ &+ C_{\phi\phi}\frac{\nu_t}{\phi}\underline{\text{grad}\phi}\cdot\underline{\text{grad}\phi} \\ &+ C_{\phi k}\frac{\nu_t}{k}\underline{\text{grad}\phi}\cdot\underline{\text{grad}k} \\ &+ C_{kk}\frac{\nu_t\phi}{k^2}\underline{\text{grad}k}\cdot\underline{\text{grad}k}\end{aligned} \quad (2)$$

The eddy viscosity is classically given by: $\nu_t = C_\mu k^2/\varepsilon$ with $C_\mu = 0.09$. ε is obtained from k and ϕ through $\phi = k^a\varepsilon^b$. The transport equations for a generic k_θ and ϕ_θ model (Daris, 2002, Daris and Bézard, 2002) are:

$$\frac{Dk_\theta}{Dt} = P_\theta - \varepsilon_\theta + \text{div}\left[\frac{\alpha_t}{\sigma_{k_\theta}}\underline{\text{grad}k_\theta} + \frac{\alpha_t}{\sigma_{k_\theta\phi_\theta}}\frac{k_\theta}{\phi_\theta}\underline{\text{grad}\phi_\theta}\right] \quad (3)$$

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$$
\begin{aligned}
\frac{D\phi_\theta}{Dt} &= \frac{\phi_\theta}{2k_\theta}\left(C_{p_1}P_\theta - C_{d_1}\varepsilon_\theta\right) + \frac{\phi_\theta}{k}\left(C_{p_2}P - C_{d_2}\varepsilon\right) \\
&+ \mathrm{div}\left[\frac{\alpha_t}{\sigma_{\phi_\theta}}\underline{\mathrm{grad}\phi_\theta} + \frac{\alpha_t}{\sigma_{\phi_\theta k_\theta}}\frac{\phi_\theta}{k_\theta}\underline{\mathrm{grad}k_\theta}\right] \\
&+ C_{\phi_\theta\phi_\theta}\frac{\alpha_t}{\phi_\theta}\underline{\mathrm{grad}\phi_\theta}\cdot\underline{\mathrm{grad}\phi_\theta} \\
&+ C_{\phi_\theta k_\theta}\frac{\alpha_t}{k_\theta}\underline{\mathrm{grad}\phi_\theta}\cdot\underline{\mathrm{grad}k_\theta} \\
&+ C_{k_\theta k_\theta}\frac{\alpha_t\phi_\theta}{{k_\theta}^2}\underline{\mathrm{grad}k_\theta}\cdot\underline{\mathrm{grad}k_\theta}
\end{aligned}
\qquad (4)
$$

The turbulent thermal diffusivity is modelled as $\alpha_t = C_\lambda k\tau^m(2\tau_\theta)^n$ with $m+n=1$ and $C_\lambda \approx 0.11$, where $\tau = k/\varepsilon$ and $\tau_\theta = k_\theta/\varepsilon_\theta$ respectively represent the dynamic and the thermal time scales. Usually the assumption is made that τ and τ_θ have the same importance in the turbulent thermal diffusivity expression which implies that $m = n = 1/2$ following Nagano and Kim (1988).

CONSTRAINTS

Relations between the model constants can be obtained by studying several basic flow behaviours considered as essential, such as:

- the decay of homogeneous turbulence (Aupoix, 1987 and Newman et al., 1981),
- the characteristics of the logarithmic region for a zero pressure gradient (ZPG) and for a moderate APG boundary layer (Huang and Bradshaw, 1995),
- the characteristics of the square-root region for a strong APG boundary layer (Townsend, 1961),
- the independence of the turbulent flow to the outer turbulent values prescribed in the boundary conditions (Cazalbou et al., 1994).

By prescribing that the model has to reproduce these elementary behaviours, analytical expressions between the constants of the full $k-\phi$ / $k_\theta - \phi_\theta$ model are obtained, as it will be shown.

Decay of isotropic turbulence

In a grid-generated turbulence without any temperature and velocity gradient, the production and diffusion terms in equations (1) to (4) disappear and relations between C_{ϕ_2}, C_{d_1} and C_{d_2} can be derived.

By studying the kinetic energy spectra, Aupoix (1987) found that the constant C_{ϕ_2} of the destruction term of the ϕ transport equation (2) has to follow:

$$
\frac{17}{10} < \frac{C_{\phi_2} - a}{b} < 2 \qquad (5)
$$

In a similar way, by studying the variance spectra (for a Prandtl number lower than one), the following relation can be obtained:

$$
\frac{17}{5} < \frac{3}{2}\frac{C_{d_1} - 2p}{q} + R_{eq}\left[\frac{3C_{d_2}}{q} - \frac{C_{\phi_2} - a}{b}\right] < 4 \qquad (6)
$$

where R_{eq} represents the equilibrium value of the ratio R of the thermal to the dynamic time scales ($R = \tau_\theta/\tau$).

Newman et al. (1981) have shown that R reaches its equilibrium value R_{eq} in a uniform way far from the grid, which means that DR/Dt has always the same sign, and that different values of R_{eq} can be obtained depending on the grid characteristics as seen in Warhaft and Lumley (1978) experiments. Usually models give one unique value for R_{eq}, which is not correct. Thus another condition arises that the model has to be able to give different values of R_{eq}. Deriving a transport equation for R, it can be shown that these two conditions leads to:

$$
\begin{aligned}
&\frac{2}{5} < R_{eq} < \frac{5}{2}\ ; \quad \frac{C_{d_2}}{q} + 1 = \frac{C_{\phi_2} - a}{b} \\
&C_{d_2} \geq 0\ ; \quad C_{d_1} = 2(p+q)
\end{aligned}
\qquad (7)
$$

The limits on R_{eq} given in (7) are in agreement with Warhaft and Lumley (1978) experiments.

Logarithmic region

In the logarithmic region of a ZPG boundary layer, the production of k_θ equilibrates the dissipation rate ε_θ and the turbulent Prandtl number is close to 0.85. The temperature profile has a logarithmic evolution with the wall distance and the temperature gradient can be written as:

$$
\frac{\partial T^+}{\partial y^+} = \frac{1}{\kappa_{t_0} y^+} \quad \text{with } T^+ = \frac{T_w - T}{T_\tau} \qquad (8)
$$

where the subscript $^+$ indicate wall quantities, T_w is the wall temperature, $T_\tau = \Phi_w/(\rho C_p u_\tau)$ is the friction temperature and $\kappa_{t_0} = \kappa_0/Pr_t = 0.48$ is the thermal equivalent of the von Kármán constant for the velocity profile $\kappa_0 = 0.41$. Using the transport equations of k_θ and ϕ_θ, κ_{t_0} can be obtained through the following relation:

$$
\frac{\sqrt{C_\mu}}{q^2\kappa_{t_0}^2 Pr_t}\left(\frac{C_{d_1} - C_{p_1}}{2R} + C_{d_2} - C_{p_2}\right) = C_{\phi_\theta\phi_\theta} + \frac{1}{\sigma_{\phi_\theta}} \qquad (9)
$$

When a moderate pressure gradient is applied, the temperature profile has no longer a logarithmic evolution with the wall distance. However Huang and Bradshaw (1995) have shown that the turbulent Prandtl number remains constant and equal approximately to 0.85, an assumption which is based upon experimental data. All equations and expressions can be developed in terms of p^+y^+, where p^+ is the normalized pressure gradient ($p^+ = \nu/\rho u_\tau^3 dp/dx$). The von Kármán constant κ_t for the temperature profile ($\partial T^+/\partial y^+ = 1/\kappa_t y^+$) is expanded to the first order in p^+y^+:

$$
\kappa_t = \kappa_{t_0} + \kappa_{t_1} p^+ y^+ \qquad (10)
$$

Expanding the expression of the turbulent Prandtl number to first order in p^+y^+ leads to:

$$
Pr_t = \frac{\kappa_0}{\kappa_{t_0}}\left[1 + \left(1 + \frac{\kappa_1}{\kappa_0} - \frac{\kappa_{t_1}}{\kappa_{t_0}}\right)p^+y^+\right] \qquad (11)
$$

where κ_1 is the dynamic counterpart of κ_{t_1} for the velocity profile. According to experiments $\kappa_1 = 0$, which means that the assumption of a constant turbulent Prandtl number with the pressure gradient leads to: $\kappa_{t_1}/\kappa_{t_0} = 1$. Then integrating (10), Daris (2002) has shown that the temperature profile is:

$$
T^+ = \frac{1}{\kappa_{t_0}}\ln(\xi) + \text{cste} \quad \text{with} \quad \xi = \frac{y^+}{1 + p^+y^+} \qquad (12)
$$

Using the ξ variable, the temperature profile in APG boundary layer flows takes the classical logarithmic shape.

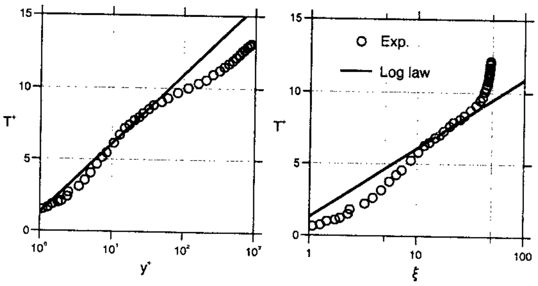

Figure 1: Evidence of the logarithmic law on the temperature profile for an APG boundary layer (Orlando et al., 1974).

This transformation is validated in Fig. 1 using the experimental data of Orlando et al. (1974) for an APG boundary layer. On the left part of the figure the temperature profile is plotted classically in wall variables and does not follow the logarithmic law. On the right part the temperature profile is plotted using the ξ variable and the logarithmic law appears naturally.

Square root region

For a strong APG boundary layer flow ($p^+y^+ \gg 1$), Townsend (1961) has shown that the velocity profile exhibits a square root evolution with the wall distance:

$$U^+ = \frac{2}{\kappa_0}\sqrt{p^+y^+} + \text{cst} \tag{13}$$

This square-root evolution can be confirmed by experimental data. Using again the Huang and Bradshaw (1995) assumption of a constant turbulent Prandtl number with the pressure gradient, the temperature equation leads to the following evolution with the wall distance:

$$T^+ = -\frac{2Pr_t}{\kappa_0}\frac{1}{\sqrt{p^+y^+}} + \text{cst} \tag{14}$$

All transport equations can be developed assuming p^+y^+ is much greater than unity and the theoretical behaviour of the velocity and the temperature profiles given by the model in the square root region can be found as relations between the model constants. However these relations are too complex to be presented in this paper.

Turbulent edge behaviour

At the outer edge of the turbulent flow, there is a balance between convection and diffusion, which means that the production and the destruction terms of equations (1) to (4) have to decrease more rapidly than the other terms. Moreover it can be shown experimentally that the temperature and the turbulent quantities k_θ and ε_θ go to the edge values very smoothly. This behaviour can be expressed as constraints on the model constant. The non-dimensional flow quantities $\tilde{T}$, $\tilde{k}$, etc. are assumed to evolve as a power of the non-dimensional distance to the edge $\lambda = 1 - y/\delta$ (where δ is a characteristic thickness) as follows:

$$\begin{aligned}\tilde{T} &= \frac{T - T_e}{T_r} = T_\theta \lambda^{e_T} \; ; \; \tilde{k}_\theta = \frac{k_\theta}{T_r^2} = K_\theta \lambda^{e_{k_\theta}} \\ \tilde{\varepsilon}_\theta &= \frac{\varepsilon_\theta \delta}{U_r T_r^2} = E_\theta \lambda^{e_{\varepsilon_\theta}} \; ; \; \tilde{\phi}_\theta = \Phi_\theta \lambda^{e_{\phi_\theta}}\end{aligned} \tag{15}$$

where T_e is the freestream temperature, U_r and T_r are some reference velocity and temperature. It can be shown

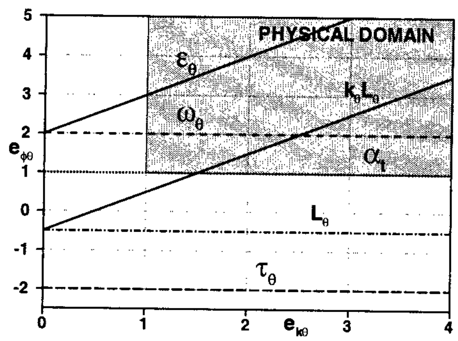

Figure 2: Theoretical behaviour of different thermal scales at the edge. $e_k = 3$.

straightforwardly from the definition of ϕ_θ that $\Phi_\theta = K_\theta{}^p E_\theta{}^q$ and $e_{\phi_\theta} = pe_{k_\theta} + qe_{\varepsilon_\theta}$. Saying that the quantities have to go to the edge smoothly (when $y \to \delta$ or $\lambda \to 0$) is equivalent to say that: $e_T > 1$, $e_{k_\theta} > 1$ and $e_{\phi_\theta} > 1$. Cazalbou et al. (1994) have shown that these physical constraints ensure the independence of the solution to the residual level of k_θ and ϕ_θ to be prescribed in the outer boundary conditions. It should be emphasized that the constraints concern the transported quantities, which means that k_θ and ε_θ can have the correct behaviour whereas ϕ_θ not, depending on the choice of the scale. Indeed, from the definition of the thermal diffusivity α_t, it can be shown that:

$$e_{\phi_\theta} = (p+q)e_{k_\theta} + q(e_k - 1) \tag{16}$$

where e_k characterises the evolution of k at the edge. As for k_θ, a physical behaviour of k is obtained if $e_k > 1$. Knowing p, q, e_k and e_{k_θ}, the behaviour of ϕ_θ can be obtained. Fig. 2 presents the evolution of e_{ϕ_θ} with e_{k_θ} for different scales with $e_k = 3$ which ensures a smooth evolution of k at the edge.

The physical domain for the turbulent scales (i.e. $e_{k_\theta} > 1$ and $e_{\phi_\theta} > 1$) is shaded in the figure. It can be seen that some scales cannot behave correctly like the thermal time scale τ_θ or the thermal length scale L_θ as the power of their evolution with λ is negative, which means that they tend to infinity at the edge. It has been checked that this behaviour is independent of the value chosen for e_k. Thus a turbulence model based on τ_θ or L_θ will depend on the residual value prescribed outside. The thermal diffusivity α_θ is on the limit as its power is just equal to unity. Other scales like ε_θ, ω_θ or $k_\theta L_\theta$ can behave well at the edge. However the powers e_{k_θ} and e_{ϕ_θ} of a given model will depend on the choice of the different constant values and the final behaviour could be unphysical even with favorable scales. The behaviour of the dynamic scales is similar: the time scale τ and the length scale L cannot behave well at the edge, whereas ε, ω or kL can be used and the eddy viscosity ν_t is on the limit.

To satisfy the balance between convection and diffusion (and extra diffusion terms), the powers of T and k_θ have to fulfill: $2e_T > e_{k_\theta}$. By equating the same powers of λ in the k_θ and ϕ_θ equations, the following relations can be found:

$$\begin{aligned}&\frac{e_{k_\theta}}{\sigma_{k_\theta}} + \frac{e_{\phi_\theta}}{\sigma_{k_\theta\phi_\theta}} = e_T \\ &e_{\phi_\theta}^2\left[C_{\phi_\theta\phi_\theta} + \frac{1}{\sigma_{\phi_\theta\phi_\theta}}\right] + e_{\phi_\theta}e_{k_\theta}\left[C_{\phi_\theta k_\theta} + \frac{1}{\sigma_{\phi_\theta k_\theta}}\right] \\ &\quad - e_{\phi_\theta}e_T + C_{k_\theta k_\theta}e_{k_\theta}^2 = 0\end{aligned} \tag{17}$$

Finally it can be assumed that for a fully-developed turbulent flow, the dynamic and thermal structures, and so the

Table 1: Behaviour of existing $k - \varepsilon$ / $k_\theta - \varepsilon_\theta$ models in the logarithmic region.

Constraint	NK	HNT	SSZ	Exp. or theory
κ_0	0.42	0.42	0.385	0.41
κ_1/κ_0	2.4	1.7	3.3	0
κ_{t_0}	0.44	0.43	0.48	0.48
$\kappa_{t_1}/\kappa_{t_0}$	-1.33	-2.64	11.5	1

dynamic and the thermal boundaries, are the same (Orlando et al., 1974). This means that the eddy viscosity and thermal diffusivity are similar at the edge and that the ratio, i.e. the turbulent Prandtl number $Pr_t = \nu_t/\alpha_t$, tends to unity.

ANALYSIS OF EXISTING $K_\theta - \varepsilon_\theta$ MODELS

Famous four-equation models are the $k - \varepsilon$ / $k_\theta - \varepsilon_\theta$ models of Nagano and Kim (1988), Hattori et al. (1993) and Sommer et al. (1993), which are respectively noted NK, HNT and SSZ. The corresponding transport equations are simpler than the generic equations (1) to (4) as they have no extra-diffusion terms ($\sigma_{k\phi} = \sigma_{\phi k} = \sigma_{k_\theta \phi_\theta} = \sigma_{\phi_\theta k_\theta} = \infty$) and no cross terms ($C_{\phi\phi} = C_{\phi k} = C_{kk} = C_{\phi_\theta \phi_\theta} = C_{\phi_\theta k_\theta} = C_{k_\theta k_\theta} = 0$).

With regards to homogeneous flows, only the HNT model satisfies all the relations (5) to (7). NK and SSZ models give unique values for R_{eq} (respectively of 1 and 10/3) and thus cannot predict the different values of R_{eq} of Warhaft and Lumley (1978) experiments.

The behaviour in APG logarithmic region is summarised in Tab. 1. As expected all models give almost $\kappa_0 = 0.41$ and $\kappa_{t_0} = 0.48$, but none of them is able to give the correct value of 0 for κ_1 and 1 for the ratio $\kappa_{t_1}/\kappa_{t_0}$. This means none of them is able to predict the correct behaviour in APG flows. It comes initially from the bad behaviour of the dynamic part of the models which are $k - \varepsilon$ models. However it can be shown that assuming a perfect behaviour of the dynamic models (i.e. assuming $\kappa_1 = 0$), the behaviour of the $k_\theta - \varepsilon_\theta$ models is slightly improved but still stays uncorrect.

In strong APG flows, when trying to solve the equations assuming high values of p^+y^+, no square-root solution for U and T can be found. This means that none of the models is able to predict the square-root region of the velocity and temperature profiles.

For the behaviour at the edge, relations (17) simplify as:

$$e_T = \frac{e_k - 1}{\sigma_{\varepsilon_\theta} - \sigma_{k_\theta}} \; ; \; e_{k_\theta} = \sigma_{k_\theta} e_T \; ; \; e_{\phi_\theta} = \sigma_{\varepsilon_\theta} e_T \qquad (18)$$

and the constraints $e_T > 1$, $e_{k_\theta} > 1$ and $e_{\varepsilon_\theta} > 1$ become:

$$\begin{aligned} &\sigma_{k_\theta} < 2 \; ; \; \sigma_{k_\theta} < \sigma_{\varepsilon_\theta} \\ &\sigma_{k_\theta}/\sigma_{\varepsilon_\theta} < e_k \; ; \; 1 + \sigma_{\varepsilon_\theta} - \sigma_{k_\theta} < e_k \end{aligned} \qquad (19)$$

Only the SSZ model is able to behave physically. It predicts the following powers for the evolution of the physical quantities: $e_T = 20$, $e_{k_\theta} = 15$ and $e_{\varepsilon_\theta} = 30$, which is far above the limit of 1 and provide a very smooth evolution at the edge. With SSZ model, the turbulent Prandtl number tends to unity, as it should be.

For NK and HNT models, the diffusion constants σ_{k_θ} and $\sigma_{\varepsilon_\theta}$ are equal, which gives a singularity in the solution at the edge as seen in expression (18). Therefore these models predict: $e_T = e_{k_\theta} = e_{\varepsilon_\theta} = +\infty$ for NK model and $-\infty$ for HNT model, and the same for the limit of the turbulent Prandtl number at the edge.

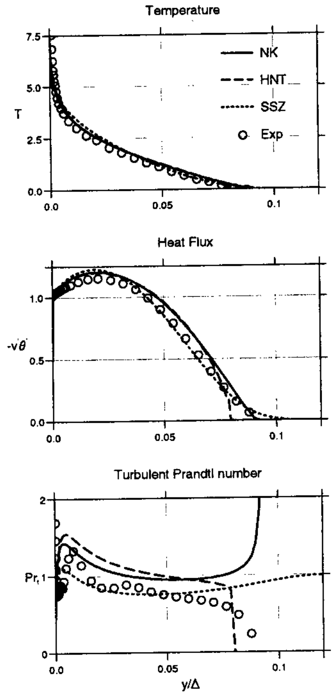

Figure 3: Comparison of $k_\theta - \varepsilon_\theta$ models with experiments (Orlando et al., 1974) for a heated APG boundary layer ($p^+ \approx 0.02$).

These theoretical results have been validated on the APG heated boundary layer test case of Orlando et al. (1974). Only the outer part of the boundary layer is computed in this case to get rid of any near-wall model. The numerical resolution is performed through a simple self-similar solver where grid convergence has been checked. The temperature, heat flux and turbulent Prandtl number profiles with the wall distance are presented in Fig. 3. All quantities are scaled by outer variables.

All models give quite good results on the main part of the flow except at the edge where models behaves exactly as predicted in the theory. This is particularly striking on the turbulent Prandtl number which goes to $+\infty$ for NK model, to $-\infty$ for HNT model and to unity for SSZ model.

THE K-KL / K_θ-$K_\theta L_\theta$ MODEL

As seen previously, no existing model is able to fulfill all prescribed constraints. Following the approach presented in this paper, a full four-equation model has been developed. For the dynamic part the method can be found in details in Catris and Aupoix (2000).

The generic transport equations (1) to (4) contains lots of degrees of freedom and some simplifications have been done. The idea was to suppress most of the cross terms which could provide numerical problems outside the turbulent flow when all quantities are close to zero, while keeping the constraints fulfilled. Both extra-diffusion terms (involving $\sigma_{k_\theta \phi_\theta}$ and $\sigma_{k_\theta \phi_\theta}$ constants) were set to zero and the cross term involving $C_{k_\theta k_\theta}$ constant was suppressed. The same choices were made for the dynamic model.

The behaviour at the edge is given by equation (17) which is of second order and admits two possible solutions. It is not possible to determine which solution will be obtained. A way to get the right evolution is to force one solution to be physical and the other not. It can be shown (Daris, 2002) that a practical way to ensure this behaviour on the dynamic model $k-\phi$ ($\phi = k^a \varepsilon^b$) is to have:

$$\frac{b}{a+2b} < 0 \tag{20}$$

The $k-kL$ model with $L = k^{3/2}/\varepsilon$ fits this constraint (a=5/2, b=-1). The kL scale is also interesting because it goes naturally to zero at the wall, which greatly simplifies the wall-boundary condition, and is linear in the logarithmic region of a ZPG boundary layer (as k is constant and L is linear), which should provide a relative independence to the mesh size in this region. The unknown constants of the $k-kL$ model were set using the relations coming from the dynamic constraints so that the $k-kL$ model used here fulfills all constraints.

By analogy, a $k_\theta - k_\theta L_\theta$ thermal model was chosen, where L_θ is the thermal length scale and can be expressed as $L_\theta = \sqrt{\varepsilon}(k_\theta/\varepsilon_\theta)^{3/2}$ (Yoshizawa, 1988). It can be noticed that the $k_\theta L_\theta$ scale cannot be exactly represented with $k_\theta{}^p \varepsilon_\theta{}^q$ as ε enters in its definition. However the approach presented here can be easily extended to this scale. As the dynamic kL scale, the $k_\theta L_\theta$ scale goes naturally to zero at the wall and is linear in the logarithmic region of a ZPG boundary layer. The determination of the exact transport equation of the $k_\theta - k_\theta L_\theta$ model can be done in a similar way as for the $k-kL$ model (Wolfshtein, 1969) through the study of the two-point temperature fluctuations, which leads to suppress the k-production term in the $k_\theta L_\theta$ transport equation (involving constant C_{p_2}). Finally the set of equations solved in the case of an incompressible high Reynolds number flow are:

$$\frac{Dk}{Dt} = P - \varepsilon + \operatorname{div}\left[\frac{\nu_t}{\sigma_k}\underline{\operatorname{grad}k}\right] \tag{21}$$

$$\begin{aligned}\frac{D\phi}{Dt} &= \frac{\phi}{k}\left(C_{\phi_1}P - C_{\phi_2}\varepsilon\right) + \operatorname{div}\left[\frac{\nu_t}{\sigma_\phi}\underline{\operatorname{grad}\phi}\right] \\ &+ C_{\phi\phi}\frac{\nu_t}{\phi}\underline{\operatorname{grad}\phi}\cdot\underline{\operatorname{grad}\phi} + C_{\phi k}\frac{\nu_t}{k}\underline{\operatorname{grad}\phi}\cdot\underline{\operatorname{grad}k}\end{aligned} \tag{22}$$

$$\frac{Dk_\theta}{Dt} = P_\theta - \varepsilon_\theta + \operatorname{div}\left[\frac{\alpha_t}{\sigma_{k_\theta}}\underline{\operatorname{grad}k_\theta}\right] \tag{23}$$

$$\begin{aligned}\frac{D\phi_\theta}{Dt} &= \frac{\phi_\theta}{2k_\theta}\left(C_{p_1}P_\theta - C_{d_1}\varepsilon_\theta\right) - C_{d_2}\frac{\varepsilon\phi_\theta}{k} \\ &+ \operatorname{div}\left[\frac{\alpha_t}{\sigma_{\phi_\theta}}\underline{\operatorname{grad}\phi_\theta}\right] + C_{\phi_\theta\phi_\theta}\frac{\alpha_t}{\phi_\theta}\underline{\operatorname{grad}\phi_\theta}\cdot\underline{\operatorname{grad}\phi_\theta} \\ &+ C_{\phi_\theta k_\theta}\frac{\alpha_t}{k_\theta}\underline{\operatorname{grad}\phi_\theta}\cdot\underline{\operatorname{grad}k_\theta}\end{aligned} \tag{24}$$

$$\begin{aligned}\phi &= kL = \frac{k^{5/2}}{\varepsilon} \; ; \; \nu_t = C_\mu\frac{\phi}{\sqrt{k}} \\ \phi_\theta &= k_\theta L_\theta = \frac{k_\theta{}^{5/2}}{\varepsilon_\theta}\sqrt{\frac{\varepsilon}{\varepsilon_\theta}} \; ; \; \alpha_t = 2^{3/2}C_\lambda\frac{\phi_\theta}{\sqrt{k_\theta}}\sqrt{\frac{k}{k_\theta}}\end{aligned} \tag{25}$$

The model constants resulting from all constraints are listed in Tab. 2.

APPLICATIONS

The $k_\theta - k_\theta L_\theta$ model has been applied on Orlando et al. (1974) experiments of a slightly heated APG boundary layer. The dynamic field is computed with the dynamic $k-kL$ model. The comparison on the non-dimensional temperature and heat flux profiles presented in Fig. 4 is fairly good. It can be noticed the smooth shape of the profiles at the edge as prescribed. The turbulent Prandtl number does not behave as expected, especially at the edge where the limit should be equal to one. However this result should be considered with caution as convergence was not fully obtained on this variable. Numerical problems arise in the self-similar solver, especially outside the turbulent flow where all quantities are close to zero, as no natural viscosity exists. However some tests performed on the dynamic $k-kL$ model with the ONERA Navier-Stokes solver (*elsA*) did not exhibit such numerical problems.

Another application of the $k_\theta - k_\theta L_\theta$ model is presented in Fig. 5 for the heated plane far wake experiments of Antonia and Browne (1986). The SSZ $k_\theta - \varepsilon_\theta$ model is also shown for comparison. The best comparison on the temperature and heat flux profiles is obtained with the $k_\theta - k_\theta L_\theta$ model. As for the APG boundary layer flow (see Fig. 4) the profiles are very smooth at the edge of the flow, as with SSZ model. Indeed both models respect the edge behaviour and provide a high value of the power e_T of the temperature evolution. The turbulent Prandtl number given by the $k_\theta - k_\theta L_\theta$ model is always far from experiments due again to a partial numerical convergence on this quantity, whereas the SSZ model gives a good comparison with the theoretical

Table 2: $k - kL$ / $k_\theta - k_\theta L_\theta$ model constants.

C_{ϕ_1}	C_{ϕ_2}	$C_{\phi\phi}$	$C_{\phi k}$	σ_k	σ_ϕ	C_μ
1	0.58	-1.72	0.96	1.8	1.03	0.09

C_{p_1}	C_{d_1}	C_{d_2}	$C_{\phi_\theta\phi_\theta}$	$C_{\phi_\theta k_\theta}$	σ_{k_θ}	σ_{ϕ_θ}	C_λ
1	1	-0.42	-3.5	1.11	1	0.35	0.105

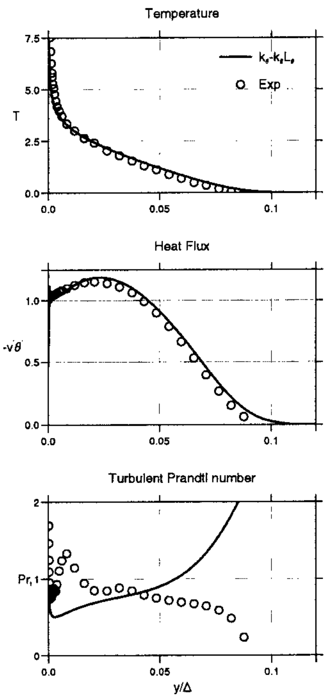

Figure 4: Comparison of $k_\theta - k_\theta L_\theta$ and $k_\theta - \varepsilon_\theta$ models with experiments (Orlando et al., 1974) for a heated APG boundary layer ($p^+ \approx 0.02$).

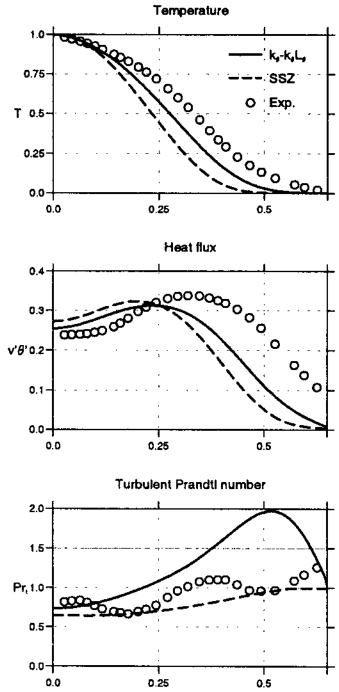

Figure 5: Comparison of $k_\theta - k_\theta L_\theta$ model and NK, HNT and SSZ $k_\theta - \varepsilon_\theta$ models with experiments (Antonia and Browne, 1986) for a heated plane far wake.

limit of one at the edge.

CONCLUSIONS

A novel approach has been developed to analyse existing thermal turbulence models or to build fully new models that are able to respect basic physical behaviours encountered in homogeneous flows, APG boundary layer flows and at the edge of turbulent flows. It leads to analytical constraints that the model constants have to fulfill. Tests performed with existing $k - \varepsilon$ / $k_\theta - \varepsilon_\theta$ models proved that the numerical solutions exhibit the behaviours predicted by the theory. A new $k - kL$ / $k_\theta - k_\theta L_\theta$ model has been developed which respect all prescribed behaviours. Preliminary tests in simple self-similar flows give satisfactory results, although the prediction of the turbulent Prandtl number suffers of lack of convergence.

Future developments will concern the implementation of the full $k - kL$ / $k_\theta - k_\theta L_\theta$ in the ONERA Navier-Stokes solver (*elsA*) and applications for more complex engineering flows with heat transfer. Moreover a near-wall model is under development that gives the linear evolution of kL and $k_\theta L_\theta$ from the wall to the logarithmic region so that the model should not depend much on the wall grid refinement.

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