RIGOROUS BOUNDS ON THE BUOYANCY FLUX IN TURBULENT STABLY STRATIFIED COUETTE FLOW

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ABSTRACT

We derive a rigorous upper bound for the long time averaged vertical buoyancy flux for stably stratified Couette flow: i.e. the flow of a Boussinesq fluid confined between two parallel horizontal plates, which are maintained at a constant (statically stable) temperature difference and driven at a constant relative velocity. We find that the upper bound $\mathcal{B}_{\max}$ does not depend on the overall or bulk Richardson number of the flow. We show that $\mathcal{B}_{\max}$ has the same characteristic scaling as the mechanical energy dissipation rate. This implies that the mixing efficiency (or flux Richardson number) is independent of both Reynolds number and bulk Richardson number for such optimal flows, or equivalently, that there is a generic partition of time-averaged turbulent dissipation of kinetic energy and vertical buoyancy flux.

I. INTRODUCTION

Flows where both the mean horizontal velocity and mean density distributions vary with height, (i.e. stably stratified shear flows) are ubiquitous in the environment. A particularly important question is how, and how much turbulent, inherently small scale motions cause "mixing", and so irreversibly modify the density distribution. Such irreversible mixing processes lead to transport of heat and/or salinity within the atmosphere or ocean. The parameterization of such transport is at the heart of the appropriate modelling of heat, salinity and momentum budgets in large scale models of geophysical flows. There has been a wide range of research trying to gain a detailed understanding of mixing within stratified shear flows in general (see, for example, Fernando, 1991; Peltier and Caulfield, 2003). It would clearly be useful if constraints or bounds could be found for the rate of mixing, (or equivalently the long time averaged buoyancy flux) within stratified shear flows, and that is the principal objective of the research which we report here.

Previous attempts to construct bounds on mixing have focussed on using semi-empirical scalings and closure assumptions for the turbulence within a stratified shear flow, in particular through heuristic relationships between the buoyancy flux and appropriate averages of the turbulent kinetic energy and density fluctuations within the flow (Townsend, 1958; Monin and Yaglom, 1971; Turner, 1973). Our research reported here (see Caulfield and Kerswell, 2001 for a fuller description) attempts to consider the same problem in a completely different, and fundamentally rigorous manner. We use the variational "background formulation" due to Doering and Constantin (see Doering and Constantin, 1992, 1994, 1996; Constantin and Doering 1995). This formulation critically relies on an insight due to Hopf (1941), and so, following Plasting and Kerswell (2003), it seems appropriate to refer to the method as the CDH method. The CDH method uses a non-unique decomposition of both the velocity and density distributions into a steady "background" that satisfies the actual inhomogeneous boundary conditions of the flow, and a "fluctuation" away from this background with homogeneous boundary conditions. (It is important to stress that the background should not be confused with the horizontal average of the flow.)

To develop rigorous bounds on the long time averaged buoyancy flux, we consider a simple model flow: namely plane Couette stratified shear flow. In this project we aim to generate two distinct bounds. As reported here, (and also, in a different way, in Caulfield and Kerswell, 2001) we generate the first bound analytically by restricting our consideration to a simple class of possible background flows consisting of piecewise linear profiles. We aim to generate the second bound numerically as a solution of the full one dimensional problem, generalizing to stratified flows the methods discussed in Plasting and Kerswell (2003), using appropriate continuation techniques. This paper is organized as follows. In Section II, we discuss the flow under consideration, and formulate the CDH problem. In Section III, we present the first bound generated by restriction of our consideration to piecewise linear profiles. In Section IV, we draw some conclusions, and briefly signpost preliminary calculations towards determining the second, better bound for general background profiles.

II. PROBLEM FORMULATION

We consider stably stratified Couette flow. We consider a layer of incompressible viscous fluid that is sheared between two infinite parallel plates located at $z = \pm\frac{1}{2}d$ which are moving with velocities $\mp\frac{1}{2}\Delta U\hat{\mathbf{x}}$ respectively. Stable stratification is enforced within the flow by maintaining the plates at constant but different temperatures so that there is a *stable* density difference across the layer of $\Delta\rho$: see figure 1. We

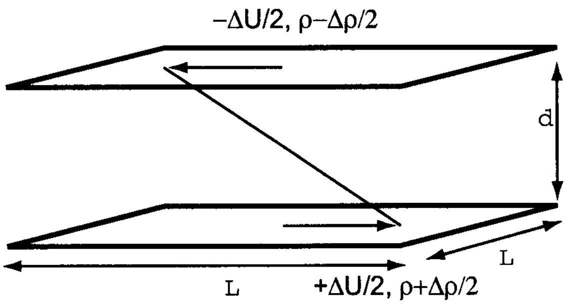

Figure 1: Schematic flow geometry, with the imposed (constant) boundary conditions.

adopt the Boussinesq approximation and use the plate separation d, a characteristic density ρ_0 (where $\Delta\rho/\rho_0 \ll 1$) and the diffusive timescale d^2/κ (κ is the thermal diffusivity) to non-dimensionalize the governing system of equations, which become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} + \nabla p - \sigma\nabla^2\mathbf{u} + \sigma^2 Re^2 J\rho\hat{\mathbf{z}} = 0 \ (\mathcal{N}) \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\ .\nabla\rho - \nabla^2\rho = 0 \ (\mathcal{D}) \quad (2)$$

$$\nabla.\mathbf{u} = 0 \quad (3)$$

where ρ is the (nondimensional) difference from ρ_0 (scaled by $\Delta\rho$) of the density. The boundary conditions are now

$$\mathbf{u}\left(x, y, \pm\frac{1}{2}, t\right) = (u_1, u_2, u_3) = \mp\left(\sigma\frac{Re}{2}\right)\hat{\mathbf{x}} \quad (4)$$

$$\rho\left(x, y, \pm\frac{1}{2}, t\right) = \mp\frac{1}{2} \quad (5)$$

The control parameters of this system are thus the Reynolds number Re, the Prandtl number σ and the (bulk) Richardson number J, defined as:

$$Re = \frac{\Delta U d}{\nu} \quad \sigma = \frac{\nu}{\kappa} \quad J = \frac{g\Delta\rho d}{\rho_0(\Delta U)^2} \quad (6)$$

In these expressions, g is the acceleration due to gravity and ν is the kinematic viscosity.

We define appropriate averages of a spatially varying quantity q as

$$\langle q(x,y,z)\rangle = \int_{-1/2}^{1/2} \overline{q} dz \quad (7)$$
$$:= \int_{-1/2}^{1/2}\left(\lim_{L\to\infty}\frac{1}{4L^2}\int_{-L}^{L}\int_{-L}^{L} q dy dx\right) dz$$

As discussed more fully in Caulfield and Kerswell (2001), provided the velocity and density fields are initially square integrable, straightforward manipulation of the governing equations yield certain balances:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \langle\|\nabla\mathbf{u}\|^2\rangle + \sigma Re^2 J\langle\rho u_3\rangle \quad (8)$$
$$+\frac{\sigma Re}{2}\left[\left.\frac{\partial\bar{u}}{\partial z}\right|_{z=1/2} + \left.\frac{\partial\bar{u}}{\partial z}\right|_{z=-1/2}\right] d\tilde{t} = 0$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \langle|\nabla\rho|^2\rangle \quad (9)$$
$$+\frac{1}{2}\left[\left.\frac{\partial\bar{\rho}}{\partial z}\right|_{z=1/2} + \left.\frac{\partial\bar{\rho}}{\partial z}\right|_{z=-1/2}\right] d\tilde{t} = 0$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t 1 + \langle\rho u_3\rangle \quad (10)$$
$$+\frac{1}{2}\left[\left.\frac{\partial\bar{\rho}}{\partial z}\right|_{z=1/2} + \left.\frac{\partial\bar{\rho}}{\partial z}\right|_{z=-1/2}\right] d\tilde{t} = 0$$

i.e. kinetic energy balance, entropy flux balance, and potential energy balance respectively, where

$$\|\nabla\mathbf{u}\|^2 = \sum_{i=1}^{3}|\nabla u_i|^2 \quad (11)$$

Although we are interested in maximizing averaged buoyancy flux $\mathcal{B}$:

$$\mathcal{B} := \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma^2 Re^2 J\langle\rho u_3\rangle\, d\tilde{t} \quad (12)$$

it is more convenient to calculate an upper bound on the purely quadratic quantity

$$\mathcal{B} + \sigma^2 Re^2 J = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma^2 Re^2 J\langle|\nabla\rho|^2\rangle\, d\tilde{t} \quad (13)$$

using (9) and (10) at fixed σ, Re and J. Therefore, we consider the Lagrangian functional $\mathcal{L}$:

$$\mathcal{L} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \left[\sigma^2 Re^2 J\langle|\nabla\rho|^2\rangle \right. \quad (14)$$
$$\left. -a\langle\mathbf{v}.(\mathcal{N})\rangle - \sigma^2 Re^2 Jb\langle\theta(\mathcal{D})\rangle\right] d\tilde{t}$$

where $(\mathcal{N})$ and $(\mathcal{D})$ are the Navier-Stokes equations and the density equation (1) and (2) respectively.

The CDH method requires the Lagrange multipliers $\mathbf{v}$ and θ to be linked to the physical fields $\mathbf{u}$ and ρ in a specific, though non-unique way:

$$\mathbf{u}(\mathbf{x},t) = \phi(z)\hat{\mathbf{x}} + \mathbf{v}(\mathbf{x},t) \quad (15)$$
$$\rho(\mathbf{x},t) = \tau(z) + \theta(\mathbf{x},t) \quad (16)$$

where the "background" fields ϕ and τ and the "fluctuation" fields $\mathbf{v}$ and θ satisfy the respective boundary conditions:

$$\phi = \mp\frac{\sigma Re}{2} \quad \tau = \mp\frac{1}{2} \quad \mathbf{v} = \mathbf{0} \quad \theta = 0 \text{ at } z = \pm\frac{1}{2} \quad (17)$$

a decomposition due to Hopf (1941).

Since $\mathbf{v}$ and θ are directly related to $\mathbf{u}$ and ρ, (14) actually imposes only mean momentum balance, total power balance, entropy flux balance and the mean heat balance (with Lagrange multipliers $-a\phi$, a, $\sigma^2 Re^2 Jb$ and $-\sigma^2 Re^2 Jb\tau$ respectively), rather than the complete Navier-Stokes equations (1) and the density equation (2). (This is a very important issue for the application of our calculations, which we return to in the last section.) Since this problem is mathematically well-posed, we drop time averages, and so wish to bound the functional

$$\mathcal{L}(a,b,\tau,\phi;\mathbf{v},\theta) = \sigma^2 Re^2 J\left\langle\tau'^2\right\rangle \quad (18)$$
$$-\mathcal{G}(a,b,\tau,\phi;\mathbf{v},\theta)$$

$$\mathcal{G} := a\sigma\left\langle\|\nabla\mathbf{v}\|^2\right\rangle \quad (19)$$
$$+(b-1)\sigma^2 Re^2 J\left\langle|\nabla\theta|^2\right\rangle$$
$$+a\left\langle v_1 v_3\phi'\right\rangle - a\sigma\left\langle\phi'' v_1\right\rangle$$
$$+\left\langle(b\tau' + a)\sigma^2 Re^2 J v_3\theta\right\rangle$$
$$-(b-2)\sigma^2 Re^2 J\left\langle\theta\tau''\right\rangle$$

where $(.)' = \frac{d}{dz}(.)$.

A critical point about the Hopf decomposition is that, if the background fields (and also, if possible the Lagrange multipliers a and b) can be chosen so that $\mathcal{L}$ is maximized over all possible fluctuation fields, then this value of $\mathcal{L}$ must correspond to a rigorous upper bound on $\mathcal{B}$, since it must be possible to construct any realizable flow $\mathbf{u}$ and ρ by appropriate choice of the fluctuation fields. If we are then able to minimize the maximum over all ϕ and τ, we then will have constructed the best possible upper bound. (See Kerswell, 1998; Plasting and Kerswell, 2003 for a further discussion of this critical point.) As mentioned in the Introduction, we are ultimately interested in the construction of two different bounds. The first bound (mainly discussed here) is conservative, in that we restrict attention to a sufficiently simple class of background and fluctuation fields so that analytical progress can be made, whereas the second bound uses numerical continuation techniques to consider all possible background and fluctuation fields.

In either case, necessary conditions for $\mathcal{L}$ to have an extremum of $\mathcal{L}$ is that $\mathcal{L}$ satisfies five Euler-Lagrange equations: with respect to the Lagrange multiplier b; the horizontally averaged fluctuation fields $\bar{v}_1$ and $\bar{\theta}$; and the meanless fluctuation fields $\hat{\mathbf{v}}$ and $\hat{\theta}$, (where $\bar{\hat{\mathbf{v}}} = \mathbf{0}$, $\bar{\hat{\theta}} = 0$) which take the form:

$$\frac{\delta\mathcal{L}}{\delta b} := -\left\langle\left|\nabla\hat{\theta}\right|^2\right\rangle - \left\langle\left(\bar{\theta}'\right)^2\right\rangle - \left\langle\tau'\hat{v}_3\hat{\theta}\right\rangle + \left\langle\bar{\theta}\tau''\right\rangle = 0 \quad (20)$$

$$\frac{\delta L}{\delta\bar{v}_1} := 2a\sigma\bar{\theta}'' + a\sigma\phi'' = 0 \quad (21)$$

$$\frac{\delta\mathcal{L}}{\delta\hat{\mathbf{v}}} := 2a\sigma\nabla^2\hat{\mathbf{v}} - a\phi'\begin{bmatrix}\hat{v}_3\\0\\\hat{v}_1\end{bmatrix} - (b\tau' + a)\sigma^2Re^2J\hat{\theta}\hat{\mathbf{z}} - a\nabla\hat{p} = \mathbf{0} \quad (22)$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\theta}} := \sigma^2Re^2J\left[2(b-1)\bar{\theta}'' + (b-2)\tau''\right] = 0 \quad (23)$$

$$\frac{\delta\mathcal{L}}{\delta\hat{\theta}} := \sigma^2Re^2J\left[2(b-1)\nabla^2\hat{\theta} - (b\tau' + a)\hat{v}_3\right] = 0 \quad (24)$$

We can solve (21) and (23) directly to yield

$$\bar{\mathbf{v}} = -\frac{1}{2}(\phi + \sigma Rez)\hat{\mathbf{x}} \quad (25)$$

$$\bar{\theta} = -\frac{(b-2)}{2(b-1)}(\tau + z) \quad (26)$$

Since, as shown in Caulfield and Kerswell (2001), the limit $b \to 1$ is of interest, we use (26) to eliminate τ from (20), eventually obtaining

$$\frac{b^2}{(b-2)}\left\langle\left(\bar{\theta}'\right)^2\right\rangle = -(b-2)\left\langle\left|\nabla\hat{\theta}\right|^2\right\rangle - a\left\langle\hat{v}_3\hat{\theta}\right\rangle \quad (27)$$

after eliminating the cubic term $\langle\bar{\theta}'\hat{v}_3\hat{\theta}\rangle$ by calculating $\langle\hat{\theta}.(24)\rangle$.

This equation can be substantially simplified by combining the conditions for entropy flux balance (9) and potential energy balance (10), which in terms of the optimal fluctuation fields can be expressed as

$$\left\langle\hat{v}_3\hat{\theta}\right\rangle = \frac{b^2\left\langle\left(\bar{\theta}'\right)^2\right\rangle}{(b-2)^2} + \left\langle\left|\nabla\hat{\theta}\right|^2\right\rangle \quad (28)$$

using the fact that the homogeneous boundary conditions for θ imply that $\langle\bar{\theta}'\rangle = 0$. Provided $\left\langle\hat{v}_3\hat{\theta}\right\rangle$ is non-zero, consistency of (28) and (27) requires that there is a very simple relationship between the two Lagrange multipliers, i.e.

$$b = (2 - a) \quad (29)$$

Therefore, (22) and (24) become

$$2a\sigma\nabla^2\hat{\mathbf{v}} - a\phi'\begin{bmatrix}\hat{v}_3\\0\\\hat{v}_1\end{bmatrix} - 2(1-a)\left[\frac{(2-a)}{a}\bar{\theta}' - 1\right]\sigma^2Re^2J\hat{\theta}\hat{\mathbf{z}} - a\nabla\hat{p} = \mathbf{0} \quad (30)$$

$$2(1-a)\nabla^2\hat{\theta} - 2(1-a)\left[\frac{(2-a)}{a}\bar{\theta}' - 1\right]\hat{v}_3 = 0 \quad (31)$$

and the functional $\mathcal{L}$ defined in (14) becomes

$$\mathcal{L} = \frac{(2-a)^2(1-a)\sigma}{a^2}\left\langle\left(\bar{\theta}'\right)^2\right\rangle + \sigma^2Re^2J + \frac{a\sigma}{4}\left\langle(\phi' + \sigma Re)^2\right\rangle - \mathcal{H}_{\phi,\tau,a}(\hat{\mathbf{v}},\hat{\theta}) \quad (32)$$

where

$$\mathcal{H}_{\phi,\tau,a}(\hat{\mathbf{v}},\hat{\theta}) := a\sigma\left\langle||\nabla\hat{\mathbf{v}}||^2\right\rangle + a\left\langle\phi'\hat{v}_1\hat{v}_3\right\rangle + (1-a)\sigma^2Re^2J \times\left[\left\langle\left|\nabla\hat{\theta}\right|^2\right\rangle - 2\left\langle\hat{v}_3\hat{\theta}\right\rangle + \frac{(2-a)}{a}\left\langle\bar{\theta}'\hat{v}_3\hat{\theta}\right\rangle\right] \quad (33)$$

For $\mathcal{L}$ to actually have an upper bound, the "spectral constraint" must be satisfied, i.e.

$$\mathcal{H}_{\phi,\tau,a}(\hat{\mathbf{v}},\hat{\theta}) \geq 0\ , \quad (34)$$

where the fluctuation fields $\hat{\theta}$ and $\hat{\mathbf{v}}$ are arbitrary, subject only to the requirements that they must satisfy homogeneous boundary conditions at $z = \pm 1/2$, and that $\hat{\mathbf{v}}$ must be incompressible. Clearly, a necessary condition for the spectral constraint to be satisfied is for $0 \leq a \leq 1$.

As is conventional in problems of this type, (30) and (31) imply that $\mathcal{H}$ is exactly zero for all fields corresponding to stationary points of $\mathcal{L}$. However, it is only at the unique maximum of $\mathcal{L}$ that the spectral constraint can be satisfied for all incompressible velocity fields $\hat{v}$ with homogeneous boundary conditions. Therefore, we can determine that the upper bound of interest is

$$\mathcal{L} \leq \mathcal{L}_{\max} := \frac{(2-a)^2(1-a)\sigma}{a^2}\left\langle\left(\bar{\theta}'\right)^2\right\rangle + \sigma^2Re^2J + \frac{a\sigma}{4}\left\langle(\phi' + \sigma Re)^2\right\rangle \quad (35)$$

where $\hat{\mathbf{v}}$ and $\hat{\theta}$ satisfy the Euler-Lagrange equations (30) and (31), and a, ϕ and $\bar{\theta}$ are chosen to minimize (35) while still satisfying the spectral constraint (34).

III. A RIGOROUS BOUND GENERATED USING PIECEWISE LINEAR PROFILES

To make analytical progress, two strong assumptions are made. Firstly, rather than trying to find non-trivial solutions to the Euler-Lagrange equations (30) and (31) for the meanless fluctuations $\hat{\mathbf{v}}$ and $\hat{\theta}$, we conservatively select the background fields to enforce these fluctuations to be exactly zero. Since any choices of ϕ, τ, (and hence $\bar{\theta}$) and a that are consistent with the spectral constraint construct a rigorous upper bound on the buoyancy flux, we will

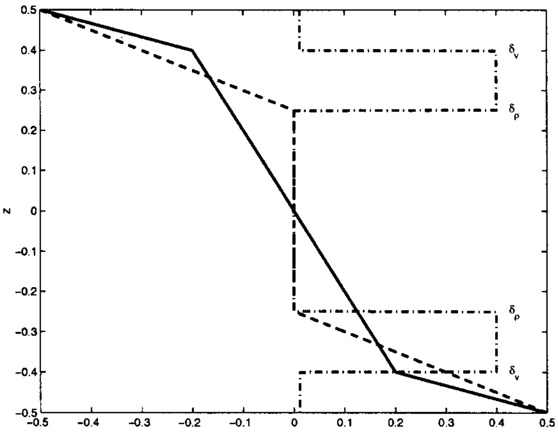

Figure 2: Mean profiles of along-stream velocity, (solid line) density, (dashed line) and gradient Richardson number (dot-dashed line) for the piecewise linear bounding solutions. (We have exaggerated the depths of the boundary layers.) Notice that the interior of the flow is well-mixed, and that the gradient Richardson number attains a very small value at the boundaries.

still construct a (conservative) rigorous bound under this assumption. Secondly, rather than minimizing $\mathcal{L}_{\max}$ over all possible a, ϕ and $\bar\theta$, we initially construct a bound using analytically tractable background profiles. Considering the physical picture of the flow shown in figure 1, a sensible strategy (favoured in previous studies by various authors: Doering and Constantin, 1992, 1996; Nicodemus et al., 1997) is to confine strong gradients to relatively thin boundary layers. Combining this physical idea with the requirement for ease of manipulation suggests the use of piecewise linear profiles, with three different regions within the flow, as shown schematically in figure 2, which shows the mean profiles in both velocity $\bar u_1 = \bar v_1 + \phi$ and density $\bar\rho = \bar\theta + \tau$.

There are linear boundary layers for both the velocity and density of depth δ_v and δ_ρ respectively, and so the gradients of ϕ and $\bar\theta$ take the form

$$\phi' = \begin{cases} -\frac{\sigma Re}{2\delta_v} & \frac12 - \delta_v < z \le \frac12 \\ 0 & |z| \le \frac12 - \delta_v \\ -\frac{\sigma Re}{2\delta_v} & -\frac12 \le z < -\frac12 + \delta_v \end{cases} \qquad (36)$$

$$\bar\theta' = \begin{cases} -\frac{a}{2(2-a)\delta_\rho} + \frac{a}{2-a} & \frac12 - \delta_\rho < z \le \frac12 \\ \frac{a}{2-a} & |z| \le \frac12 - \delta_\rho \\ -\frac{a}{2(2-a)\delta_\rho} + \frac{a}{2-a} & -\frac12 \le z < -\frac12 + \delta_\rho \end{cases} \qquad (37)$$

These fields are chosen in this form so that the depths of the boundary layers (i.e. δ_v and δ_ρ) can be varied to satisfy the spectral constraint. In particular, we choose these forms so that, throughout the interior of the flow, the sign-indefinite terms in (33) are zero.

Therefore, as discussed in detail in the work of Doering and Constantin, (1992, 1996) the sign-indefinite terms can be bounded simply using functional estimates:

$$\left\langle a\phi'\hat v_1\hat v_3\right\rangle \le \frac{a\sigma Re\delta_v}{8\sqrt{2}}\left\langle ||\nabla\hat{\mathbf{v}}||^2\right\rangle$$

$$2(1-a)\sigma^2 Re^2 J\left\langle\left[\frac{(2-a)}{a}\bar\theta' - 1\right]\hat v_3\hat\theta\right\rangle \le \frac{(1-a)\delta_\rho}{2}\left(\frac{d}{4}\left\langle ||\nabla\hat{\mathbf{v}}||^2\right\rangle + \frac{1}{d}\left\langle|\nabla\hat\theta|^2\right\rangle\right)$$

where $d > 0$ is a free positive constant. Using these expressions, we see that a sufficient condition for the spectral constraint (34) to be satisfied is that

$$(1-a)\left(1-\frac{\delta_\rho}{2d}\right) \ge 0 \qquad (38)$$

$$a - \frac{aRe\delta_v}{8\sqrt{2}} - \frac{(1-a)\sigma^2Re^2Jd\delta_\rho}{8} \ge 0 \qquad (39)$$

Clearly (38) is satisfied for $d \ge \delta_\rho/2$. We relate the depths of the two boundary layers directly by requiring the balances (8)–(10), to obtain

$$\frac{1-2\delta_v}{2\delta_v} = \frac{4J}{\sigma}\left[\frac{1-2\delta_\rho}{2\delta_\rho}\right] \qquad (40)$$

Therefore, (39) simplifies to

$$a - \frac{aRe\delta_v}{8\sqrt{2}} - \frac{\sigma Re^2J^3\delta_v^2(1-a)}{(\sigma - 2\delta_v[\sigma-4J])^2} \ge 0 \qquad (41)$$

with

$$\mathcal{L}_{\max} = \frac{\sigma^3Re^2}{4}\left[\frac{1-2\delta_v}{2\delta_v} + \frac{4J}{\sigma}\right] \qquad (42)$$

From (42) it is clear that $\mathcal{L}_{\max}$ is minimized when δ_v is maximized, which further implies that the critical situation applies when there is equality in (41). Differentiating with respect to a, we can show that δ_v attains its maximum value when $a \to 1$, and

$$\delta_v \to \frac{8\sqrt{2}}{Re} \qquad (43)$$

so that

$$\mathcal{L}_{\max} = \frac{\sigma^3Re^3}{64\sqrt{2}}\left(1-\frac{16\sqrt{2}}{Re}\right) + \sigma^2Re^2J \qquad (44)$$

Therefore, the long time averaged buoyancy flux $\mathcal{B}$ as defined in (12) is bounded by

$$\mathcal{B} \le \mathcal{B}_{\max} = \frac{\sigma^3Re^3}{64\sqrt{2}}\left(1-\frac{16\sqrt{2}}{Re}\right) \qquad (45)$$

and so the **dimensional** buoyancy flux $\hat{\mathcal{B}}$ is bounded by

$$\hat{\mathcal{B}} \le \hat{\mathcal{B}}_{\max} = \frac{(\Delta U)^3}{64\sqrt{2}d}\left(1-\frac{16\sqrt{2}}{Re}\right) \qquad (46)$$

Crucially, this bound is completely independent of the bulk stratification and diffusion within the flow, only depending on the advective scales of the velocity field. This is not entirely surprising, since the kinetic energy balance equation (8) implies that the mechanical energy dissipation and the buoyancy flux have the same scaling, and it is well-known (see Doering and Constantin, 1992; Plasting and Kerswell, 2003) that at high Re the mechanical energy dissipation scales like $\Delta U^3/d$. Also, since the bounding value of the Lagrange multiplier $a \to 1$ leads to a decoupling of the density and velocity fields, as is apparent in the expressions for $\mathcal{L}$ and $\mathcal{H}$ in (32) and (33), the buoyancy flux should cease to depend directly on J. Indeed, if $b = 2 - a = 1$, the Euler-Lagrange equation for $\bar\theta$ (23) implies that $\tau = -z$, i.e. the appropriate background density field corresponds to the laminar conductive state, and the interior of the flow should be completely well-mixed, as shown in figure 2.

It is very important to appreciate that although the upper bound on $\mathcal{B}$ is independent of J, the structure of the bounding flow itself does depend on the overall stratification. In particular, from (40) and (43), δ_ρ increases with J, so that the mixed layer in the flow interior must reduce in depth as J increases. This also means that the gradient in density near the boundaries $z = \pm\frac{1}{2}$ must decrease as J increases. This has significant implications for the structure of the gradient Richardson number $Ri(z)$, which is the natural measure of the relative significance of buoyancy and inertia (see e.g. Turner, 1973) and may be defined in terms of the nondimensional variables of this paper as

$$Ri(z) := -\sigma^2 Re^2 J \frac{d\bar{\rho}}{dz}\left(\frac{d\bar{u}_1}{dz}\right)^{-2} \tag{47}$$

We also plot a generic profile for $Ri(z)$ for the bounding profiles in figure 2. Because of the particular form of the bounding solutions, we can show that

$$\begin{aligned} Ri(\pm\tfrac{1}{2}) &= \frac{8J\delta_v^2}{\delta_\rho(1+2\delta_v)^2} \qquad (48) \\ &= \frac{16\sqrt{2}\left[\sigma\left(Re - 16\sqrt{2}\right) + 4J\right]}{(Re + 16\sqrt{2})^2} \\ &\simeq \frac{16\sqrt{2}\sigma}{Re} \quad \text{as } Re \to \infty \end{aligned}$$

establishing that the Richardson number at the boundaries is both independent of J to leading order, and also that it becomes extremely small for large Re. Therefore, as is commonly assumed for forced, steady stratified turbulence (see Turner, 1973) the flow has created regions where the stratification is extremely weak, and the density does not dynamically affect the flow evolution, thus allowing mixing to be sustained.

Similarly, it is also straightforward to calculate the dissipation of the solution that has been determined to construct an upper bound on the buoyancy flux. Using the bounding solutions, it is straightforward to show that, for piecewise profiles of the form assumed

$$\sigma\left\langle ||\nabla \mathbf{u}||^2 \right\rangle = \frac{\sigma^3 Re^2}{4}\left[\frac{1+6\delta_v}{2\delta_v}\right] \tag{49}$$

Therefore, for the bounding solutions, with boundary layer depth defined by (43),

$$\sigma\left\langle ||\nabla \mathbf{u}||^2 \right\rangle = \frac{\sigma^3 Re^3}{64\sqrt{2}}\left[1 + \frac{48\sqrt{2}}{Re}\right] \tag{50}$$

This is of particular interest, since an important quantity for parameterization of mixing within a stratified flow is the "mixing efficiency" (sometimes referred to as the "flux Richardson number" Ri_f) of the flow: the ratio of the long time average of the buoyancy flux to the sum of the buoyancy flux and the mechanical energy dissipation. This quantity is a measure of the proportion of the work done on the flow that is leading to irreversible changes in the potential energy of the fluid (see Peltier and Caulfield, 2003 for a more detailed discussion). For the bounding solution, combining (45) and (50) yields

$$Ri_f = \frac{1 - \frac{16\sqrt{2}}{Re}}{2 + \frac{32\sqrt{2}}{Re}} \tag{51}$$

which tends to 0.5 as $Re \to \infty$, implying equipartition of energy loss from the shear through dissipation and buoyancy flux for such bounding solutions.

Although this is very high compared to typical observations (see e.g. Park et al., 1994) and simulations (Peltier and Caulfield, 2003) which suggest that $Ri_f \sim 0.1-0.2$, heuristic models (Townsend, 1958; Monin and Yaglom, 1971;, Turner, 1973) have previously suggested 0.5 as an upper bound for mixing efficiency. These models also rely on the idea that there are regions of flow where the local gradient Richardson number is very small (see figure 2) which certainly occurs in the bounding solutions. Further reasons to think that there is a possibility of high mixing efficiency events have come from recent laboratory experiments and atmospheric observations of Fernando and his co-workers (Strang and Fernando, 2001; Pardyjak et al., 2002). They also observed flows, which under certain circumstances, had mixing efficiencies as high as 0.45. Therefore, it seems at least plausible that it is possible to have flows with significantly higher mixing efficiency than is commonly assumed to occur, which may have significant implications for parameterization of mixing within larger scale models using mixing efficiency estimates. (See Caulfield and Kerswell, 2001 for a more detailed discussion.)

IV. DISCUSSION AND CONCLUSIONS

The long time averaged flow that we have derived in this paper appears to make rigorous an argument for explaining the possibility of mixing that is independent of bulk stratification within the flow, and scales in the same way as the mechanical energy dissipation. This implies that, at high Reynolds number, the flux Richardson number will remain finite. Appropriately interpreted, this observation is consistent with both heuristic models, and recent observations. In particular, it suggests that conventional estimates of mixing efficiency around $0.1 - 0.2$ may be too small as a general result, and may actually be specific to the classical flows that have been considered (e.g. grid stirred turbulence as in Park et al., 1994; or KH instability break down as reviewed in Peltier and Caulfield, 2003).

However, we need to proceed with caution in such extrapolations for (at least!) five important reasons. Firstly, there is no reason to suppose that real flows actually do maximize buoyancy flux. Secondly, it is not at all clear that any results we identify from this specialized Couette flow can (or should) carry over into generic statements about stratified shear flows. Thirdly, even if we can generalize, the bound we have constructed using these piecewise linear profiles is not the best possible bound, and it is necessary to construct the best possible bound over all possible choices of background profiles, and indeed, nontrivial meanless fluctuations to draw broader conclusions. Fourthly, even if the best possible bound is generated, it is not clear that the bound is "realizable", or attained by flows that actually satisfy the Navier-Stokes equations. Due to the relationship between the fluctuation $\mathbf{v}$ and the actual fluid velocity $\mathbf{u}$, we actually optimize over a class that is a superset of the solutions to the true governing equations (1)-(3). As is well-known for the unstratified Couette flow, (see Plasting and Kerswell, 2003 for a fuller discussion) the mean flow distributions that are determined by generating an upper bound for the mechanical energy dissipation do not, in point of fact correspond to experimentally observed flows, in particular because of the continued presence of non-zero shear in the interior of the flow to asymptotically large Reynolds number. This structure (see figure 2) also persists in our stratified flow, thus calling into question whether the flows calculated here could actually be sustained in a real fluid. Finally, even if all these other problems are addressed, it is important to remember that we are only attempting to maximize the buoyancy flux. It is unclear how meaningful the associated value of the mixing efficiency for such solutions is, as the particular value of the dissipation for these optimizing solutions is not constrained in an obvious way.

It is an ongoing research activity to address all of these concerns. For example, we are actively attempting to calculate the best possible bound for this flow, by generating a second bound numerically as a solution of the full one dimensional problem numerically using appropriate continuation techniques. We aim to present the results of this calculation at the conference. To determine such a bound, it is necessary to solve the Euler-Lagrange equations (20)-(24) for variations with respect to b, $\bar{v}_1$, $\hat{\mathbf{v}}$, $\bar{\theta}$ and $\hat{\theta}$ **simultaneously** with solving the Euler-Lagrange equations for variations with respect to a, ϕ, and τ:

$$\frac{\delta \mathcal{L}}{\delta a} := -\sigma \left\langle ||\nabla \mathbf{v}||^2 \right\rangle - \left\langle \phi' v_1 v_3 \right\rangle - \sigma^2 Re^2 J \langle v_3 \theta \rangle + \sigma \left\langle \phi'' v_1 \right\rangle = 0 \quad (52)$$

$$\frac{\delta \mathcal{L}}{\delta \phi} := a\overline{v_1 v_3}' + a\sigma \overline{v_1}'' = 0 \quad (53)$$

$$\frac{\delta \mathcal{L}}{\delta \tau} := -2\tau'' + (2-a)\overline{v_3 \theta}' - a\overline{\theta}'' = 0 \quad (54)$$

using (29). The manipulation and solution of this system of equations is the topic of ongoing research.

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