TWO-POINT SIMILARITY SOLUTIONS OF SHEAR-FREE TURBULENT DIFFUSION, DIFFUSION-WAVES AND ITS IMPLICATIONS FOR RANS MODELS

Martin Oberlack, Silke Guenther
Hydromechanics and Hydraulics Group,
Darmstadt University of Technology,
Petersenstr. 13, 64287 Darmstadt, Germany
oberlack@hyhy.tu-darmstadt.de, guenther@hyhy.tu-darmstadt.de

INTRODUCTION

We reconsider the problem of shear-free turbulent diffusion with no production due to a mean-velocity gradient. Turbulence is generated at the plane $x_1 = 0$ and diffuses in the direction $x_1 > 0$. Turbulence is homogeneous in the x_2-x_3 plane. This problem was first considered by Lele (1985) raising the question whether a turbulent diffusion-wave exists by analyzing the k-ε model. In the following we show, based on the infinite sequence of multi-point correlation equations, that a variety of invariant solutions (scaling laws, see e.g. Oberlack 2001) of the diffusion problem exist employing Lie-group analysis (see e.g. Bluman & Kumei 1989).

LARGE- AND SMALL-SCALE EXPANSION OF THE MULTI-POINT CORRELATION EQUATION

For the subsequent analysis we employ the multi-point correlation (MPC) which is believed to properly model the statistical quantities of turbulence at all scales. In order to write the MPC equation in a compact form we introduce the definition

$$R_{i_{\{n+1\}}} = R_{i_{(0)}i_{(1)}\ldots i_{(n)}} = \overline{u'_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u'_{i_{(n)}}(\boldsymbol{x}_{(n)})}. \quad (1)$$

at $n+1$ points where $u'_{i_{(k)}}$ denotes velocity fluctuation about the mean velocity $\bar{u}_{i_{(k)}}$ at the point $\boldsymbol{x}_{(k)}$. With this definition at hand it is straight forward to derive the MPC equation from the Navier-Stokes equations (see Oberlack, 2000a)

$$\begin{aligned}\Theta_{i_{\{n+1\}}} = \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n}\Bigg[\bar{u}_{k_{(l)}}(\boldsymbol{x}_{(l)})\frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} \\ + R_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]}\frac{\partial \bar{u}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} \\ -\nu\frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)}\mapsto\emptyset]}\frac{\partial \overline{u'_{i_{(l)}}u'_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} \\ + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}][\boldsymbol{x}_{(n+1)}\mapsto\boldsymbol{x}_{(l)}]}}{\partial x_{k_{(l)}}} \\ + 2\Omega_k e_{i_{(l)}km}R_{i_{\{n+1\}}[i_{(l)}\mapsto m]}\Bigg] = 0 \\ \text{for} \quad n = 1,\ldots,\infty\ .\end{aligned} \quad (2)$$

by employing the additional definitions

$$R_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]} = \overline{u'_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u'_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})u'_{k_{(l)}}(\boldsymbol{x}_{(l)})u'_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)})\cdot\ldots\cdot u'_{i_{(n)}}(\boldsymbol{x}_{(n)})}, \quad (3)$$

$$R_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}][\boldsymbol{x}_{(n+1)}\mapsto\boldsymbol{x}_{(l)}]} = \overline{u'_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u'_{i_{(n)}}(\boldsymbol{x}_{(n)})u'_{k_{(l)}}(\boldsymbol{x}_{(l)})}, \quad (4)$$

$$R_{i_{\{n\}}[i_{(l)}\mapsto\emptyset]} = \overline{u'_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u'_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})u'_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)})\cdot\ldots\cdot u'_{i_{(n)}}(\boldsymbol{x}_{(n)})} \quad (5)$$

and

$$P_{i_{\{n\}}[l]} = \overline{u'_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u'_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})p'(\boldsymbol{x}_{(l)})u'_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)})\cdot\ldots\cdot u'_{i_{(n)}}(\boldsymbol{x}_{(n)})}. \quad (6)$$

The notation in square brackets denotes the replacement of certain variables or indices with some other quantities standing on the right hand side of the arrow. Each Θ-equation of the tensor order $n+1$ only contains one unclosed term of the order $n+2$. For any of the remaining terms such as $P_{i_{\{n\}}[l]}$ exact equations may be derived from the continuity equation or the Poisson equation for the pressure (see e.g. Oberlack, 2000a).

For simplicity we will proceed with the two-point correlation equation which has similar structure as the full set of multi-point correlation equations. In particular, it has the same symmetry properties. In order to simplify notation we introduce the short form

$$R_{i_{\{2\}}} = R_{ii_{(1)}} = R_{ij}. \tag{7}$$

For this case equation (2) reduces to (see e.g. Rotta, 1972)

$$\begin{aligned}
\Theta_{i_{\{2\}}} = & \frac{\bar{\mathrm{D}} R_{ij}}{\bar{\mathrm{D}} t} + R_{kj}\,\frac{\partial \bar{u}_i(\boldsymbol{x},t)}{\partial x_k} + R_{ik}\,\frac{\partial \bar{u}_j(\boldsymbol{x},t)}{\partial x_k}\bigg|_{\boldsymbol{x}+\boldsymbol{r}} \\
& + \left[\bar{u}_k\left(\boldsymbol{x}+\boldsymbol{r},t\right) - \bar{u}_k\left(\boldsymbol{x},t\right)\right]\frac{\partial R_{ij}}{\partial r_k} + \frac{\partial \overline{p'u'}_j}{\partial x_i} \\
& - \frac{\partial \overline{p'u'}_j}{\partial r_i} + \frac{\partial \overline{u'_i p'}}{\partial r_j} \\
& - \nu\left[\frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\,\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + 2\,\frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}\right] \\
& + \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right] \\
& + 2\,\Omega_k\left[e_{kli} R_{lj} + e_{klj} R_{il}\right] = 0,
\end{aligned} \tag{8}$$

where the difference between two points has been introduced according to

$$\boldsymbol{x} = \boldsymbol{x}_{(0)} \quad ,\boldsymbol{r}_{(l)} = \boldsymbol{x}_{(l)} - \boldsymbol{x}_{(0)} \quad \text{with} \quad l = 1,\ldots,n. \tag{9}$$

The vectors $\overline{p'u'}_j$ and $\overline{u'_i p'}$ are special cases of $P_{i_{\{n\}}[k]}$ defined according to

$$\begin{aligned}
\overline{p'u'}_j(\boldsymbol{x},\boldsymbol{r},t) &= \overline{p'(\boldsymbol{x}_{(0)},t)\,u'_j(\boldsymbol{x}_{(1)},t)} \ , \\
\overline{u'_i p'}(\boldsymbol{x},\boldsymbol{r},t) &= \overline{u'_i(\boldsymbol{x}_{(0)},t)\,p'(\boldsymbol{x}_{(1)},t)}.
\end{aligned} \tag{10}$$

For the two-point case the continuity equations simplify to

$$\begin{aligned}
& \frac{\partial R_{ij}}{\partial x_i} - \frac{\partial R_{ij}}{\partial r_i} = 0 \ , \quad \frac{\partial R_{ij}}{\partial r_j} = 0, \quad \frac{\partial \overline{p'u'}_i}{\partial r_i} = 0 \ , \\
& \frac{\partial \overline{u'_j p'}}{\partial x_j} - \frac{\partial \overline{u'_j p'}}{\partial r_j} = 0.
\end{aligned} \tag{11}$$

For the derivation of invariant solutions of shear-free turbulent flows below we need to investigate the symmetry properties of the multi-point correlation equations. In Oberlack (2000a) it is shown that all symmetry groups of the Euler and Navier-Stokes equations are linear transformations (see e.g. Ibragimov, 1995/1996) and hence uniquely map to a set of new symmetries for the multi-point correlation equations.

For the understanding of large Reynolds number turbulent flows it is important to note that the Euler equations admit one more scaling group compared to the Navier-Stokes equations (see e.g. Oberlack, 2000b) . It is in particular this additional scaling group which is crucial to understand turbulent scaling laws.

In order to "recover" this additional scaling group though the MPC equations contain viscosity we have to adopt these equations in a form derived from a singular asymptotic expansion first suggested in Oberlack (2000a). Therein it was proven that similar to the laminar boundary layer equations we may separate the correlation equations into an inner and outer equation corresponding to small- and large-scale turbulence. The inner equations cover the inertial range and the dissipation range. The outer equations include all large scales down to the inertial range. The inertial range is the matching region.

The following boundary layer type of expansion for small r is based on the turbulent Reynolds number

$$Re_t = \frac{\sqrt{K}\ell_t}{\nu}, \tag{12}$$

where the integral length-scale ℓ_t and the Kolmogorov length scale η_K are respectively defined as

$$\ell_t = \frac{1}{K}\int_{V_r} R_{kk}\mathrm{d}r \quad \text{and} \quad \eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \tag{13}$$

and K and ε are the turbulent kinetic energy and the dissipation of turbulent kinetic energy.

The outer part of the asymptotic expansion in r-space, i.e. the domain $\eta_K \ll r$, is obtained by taking the limit $1/Re_t \to 0$ or $\nu \to 0$ in the equations (8) yielding

$$\begin{aligned}
\Theta_{i_{\{2\}}} = & \frac{\bar{\mathrm{D}} R_{ij}}{\bar{\mathrm{D}} t} + R_{kj}\,\frac{\partial \bar{u}_i(\boldsymbol{x},t)}{\partial x_k} + R_{ik}\,\frac{\partial \bar{u}_j(\boldsymbol{x},t)}{\partial x_k}\bigg|_{\boldsymbol{x}+\boldsymbol{r}} \\
& + \left[\bar{u}_k\left(\boldsymbol{x}+\boldsymbol{r},t\right) - \bar{u}_k\left(\boldsymbol{x},t\right)\right]\frac{\partial R_{ij}}{\partial r_k} + \frac{\partial \overline{p'u'}_j}{\partial x_i} - \frac{\partial \overline{p'u'}_j}{\partial r_i} \\
& + \frac{\partial \overline{u'_i p'}}{\partial r_j} + \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right] \\
& + 2\,\Omega_k\left[e_{kli} R_{lj} + e_{klj} R_{il}\right] = 0.
\end{aligned} \tag{14}$$

It is apparent that the latter equation is not valid in the limit $r \to 0$ since no dissipation is contained which becomes important as $r \to \eta_K$.

The inner part of the asymptotic expansion of the correlation function may be obtained by introducing the singular expansion

$$\begin{aligned}
R_{ij}(\boldsymbol{x},\boldsymbol{r}) &= \overline{u'_i u'_j}(\boldsymbol{x}) - Re_t^{-\frac{1}{2}} R_{ij}^{(1)}(\boldsymbol{x},\hat{\boldsymbol{r}}) - O(Re_t^{-\frac{3}{4}}), \\
R_{(ik)j}(\boldsymbol{x},\boldsymbol{r}) &= \overline{u'_i u'_j u'_k}(\boldsymbol{x}) + Re_t^{-\frac{3}{4}} R_{(ik)j}^{(1)}(\boldsymbol{x},\hat{\boldsymbol{r}}) - O(Re_t^{-1}), \\
R_{i(jk)}(\boldsymbol{x},\boldsymbol{r}) &= \overline{u'_i u'_j u'_k}(\boldsymbol{x}) + Re_t^{-\frac{3}{4}} R_{i(jk)}^{(1)}(\boldsymbol{x},\hat{\boldsymbol{r}}) - O(Re_t^{-1}), \\
& \text{with} \quad \hat{\boldsymbol{r}} = Re_t^{\frac{3}{4}}\boldsymbol{r}
\end{aligned} \tag{15}$$

into (8) resulting in the leading order equation

$$\begin{aligned}
& \frac{\bar{\mathrm{D}}\overline{u'_i u'_j}}{\bar{\mathrm{D}} t} + \overline{u'_j u'_k}\,\frac{\partial \bar{u}_i(\boldsymbol{x},t)}{\partial x_k} + \overline{u'_i u'_k}\,\frac{\partial \bar{u}_j(\boldsymbol{x},t)}{\partial x_k} \\
& - \frac{\partial \bar{u}_k(\boldsymbol{x},t)}{\partial x_l}\,\hat{r}_l\,\frac{\partial R_{ij}^{(1)}}{\partial \hat{r}_k}\left[\frac{\partial \overline{p'u'}_j}{\partial x_i} - \frac{\partial \overline{p'u'}_j}{\partial r_i} + \frac{\partial \overline{u'_i p'}}{\partial r_j}\right]\Bigg|_{r=0} \\
& + 2\,\frac{\partial^2 R_{ij}^{(1)}}{\partial \hat{r}_k \partial \hat{r}_k} + \frac{\partial \overline{u'_i u'_j u'_k}}{\partial x_k} - \frac{\partial}{\partial \hat{r}_k}\left[R_{(ik)j}^{(1)} - R_{i(jk)}^{(1)}\right] \\
& + 2\,\Omega_k\left[e_{kli}\overline{u'_j u'_l} + e_{klj}\overline{u'_i u'_l}\right] = 0.
\end{aligned} \tag{16}$$

The pressure-velocity correlations $\overline{p'u'}_j$ and $\overline{u'_i p'}$ are determined by the Poisson equation and hence are not independent of the velocity correlations.

SYMMETRIES AND INVARIANT SOLUTIONS OF THE CORRELATION EQUATION

Since the multi-point equations of different tensor order have similar structure, in the following we solely present the two-point correlation equation. It is important to note that all results to be presented below are fully consistent with all higher order correlation functions up to infinite order.

We are primarily interested in large-scale quantities such as the Reynolds-stress tensor or the integral length-scale and hence we adopt the large-scale two-point correlation equation (14) which for the present flow of shear free diffusion reduces to

$$\frac{\partial R_{ij}}{\partial t}+\delta_{i1}\frac{\partial\overline{p'u'}_j}{\partial x_1}-\frac{\partial\overline{p'u'}_j}{\partial r_i}+\frac{\partial\overline{u'_ip'}}{\partial r_j}+\frac{\partial R_{(i1)j}}{\partial x_1}$$
$$-\frac{\partial}{\partial r_k}\left[R_{(ik)j}-R_{i(jk)}\right]+2\,\Omega_k\left[e_{kli}R_{lj}+e_{klj}R_{il}\right]=0, \tag{17}$$

extended by the kinematic conditions for the correlation functions derived from the continuity equation

$$\delta_{i1}\frac{\partial R_{1j}}{\partial x_1}-\frac{\partial R_{ij}}{\partial r_i}=0,\quad \frac{\partial R_{ij}}{\partial r_j}=0,$$
$$\frac{\partial\overline{u'_1p'}}{\partial x_1}-\frac{\partial\overline{u'_jp'}}{\partial r_j}=0,\quad \frac{\partial\overline{p'u'}_i}{\partial r_i}=0. \tag{18}$$

For a non-rotating frame of reference ($\boldsymbol{\Omega}=0$) the equations (17) and (18) admit the following classical symmetries in generator form

$$\mathrm{X}_{s_x}=x_1\frac{\partial}{\partial x_1}+r_i\frac{\partial}{\partial r_i}+2R_{ij}\frac{\partial}{\partial R_{ij}}+\dots, \tag{19}$$

$$\mathrm{X}_{s_t}=t\frac{\partial}{\partial t}-2R_{ij}\frac{\partial}{\partial R_{ij}}+\dots, \tag{20}$$

$$\mathrm{X}_{x_1}=\frac{\partial}{\partial x_1}, \tag{21}$$

$$\mathrm{X}_{t}=\frac{\partial}{\partial t}, \tag{22}$$

where dots denote additional higher order correlation functions which have been omitted.

Employing Lie's first theorem we may rewrite the symmetries in global form

$$\bar{T}_{s_x}:\ t^*=t\ ,\quad x_1^*=e^{a_1}x_1\ ,$$
$$\boldsymbol{r}^*=e^{a_1}\boldsymbol{r}\ ,\quad \mathbf{R}^*=e^{2a_1}\mathbf{R}\ ,\quad\dots \tag{23}$$
$$\bar{T}_{s_t}:\ t^*=e^{a_2}t\ ,\quad x_1^*=x_1\ ,$$
$$\boldsymbol{r}^*=\boldsymbol{r}\ ,\quad \mathbf{R}^*=e^{-2a_2}\mathbf{R}\ ,\quad\dots \tag{24}$$
$$\bar{T}_{x_1}:\ t^*=t\ ,\quad x_1^*=x_1+a_3\ ,$$
$$\boldsymbol{r}^*=\boldsymbol{r}\ ,\quad \mathbf{R}^*=\mathbf{R}\ ,\quad\dots \tag{25}$$
$$\bar{T}_{t}:\ t^*=t+a_4\ ,\quad x_1^*=x_1\ ,$$
$$\boldsymbol{r}^*=\boldsymbol{r}\ ,\quad \mathbf{R}^*=\mathbf{R}\ ,\quad\dots \tag{26}$$

which respectively correspond to scaling of space, scaling of time, translation in space and translation in time. The a_i's are the corresponding group parameter. Again dots refer to the omitted correlation functions.

Note that the equations (17) and (18) admit additional symmetries which may not be employed for the present purpose to derive scaling laws: Galilean invariance, rotation invariance about x_1 and all three reflection groups for $\boldsymbol{\Omega}=0$. If rotation about x_1 is considered the reflection groups in the x_2-x_3 plane are not admitted.

From a given set of symmetries we know from basic group theory that also any linear combination of them is a new symmetry. Hence we may combine all of the latter symmetries and rewrite the resulting symmetry in generator form

$$\mathrm{X}=a_1\mathrm{X}_{s_x}+a_2\mathrm{X}_{s_t}+a_3\mathrm{X}_{x_1}+a_4\mathrm{X}_t. \tag{27}$$

The latter combined generator may be rewritten to obtain the separated infinitesimals

$$\begin{aligned}\xi_{x_1}&=a_1x_1+a_3,\\ \xi_t&=a_2t+a_4,\\ \xi_{r_k}&=a_1r_k,\\ \eta_{R_{ij}}&=2(a_1-a_2)R_{ij},\\ \vdots\ \ &\quad\ \vdots\ .\end{aligned} \tag{28}$$

Invoking the condition of an invariant solution we obtain (see e.g. Bluman & Kumei, 1989)

$$\frac{\mathrm{d}x_1}{a_1x_1+a_3}=\frac{\mathrm{d}t}{a_2t+a_4}=\frac{\mathrm{d}r_{[k]}}{a_1r_{[k]}}=\frac{\mathrm{d}R_{[ij]}}{2(a_1-a_2)R_{[ij]}}=\dots, \tag{29}$$

where indices in brackets indicate no summation. Depending on the scaling group parameter a_1 and a_2 we distinguish three different cases. In the following we employ the abbreviation

$$m=\frac{a_2}{a_1}-1. \tag{30}$$

Turbulent diffusion with spatially growing integral length-scale ($a_1\neq 0$, $a_2\neq 0$)

Integration of (29) leads to a set of invariants which are taken as the new independent and dependent variables

$$\tilde{x}_1=\frac{x_1+x_o}{(t+t_o)^{1/(m+1)}},\quad \tilde{\boldsymbol{r}}=\frac{\boldsymbol{r}}{x_1+x_o},$$
$$R_{ij}(x_1,t,\boldsymbol{r})=(x_1+x_o)^{-2m}\tilde{R}_{ij}(\tilde{x}_1,\tilde{\boldsymbol{r}}),\ \dots, \tag{31}$$

where here and in the following subsections the constants x_o and t_o are combinations of the a_i's. The key achievement is that the variables (31) lead to a similarity reduction of (17)/(18). From (13) and by invoking the one-point limit in (31) we obtain

$$\overline{u'_iu'_j}(x_1,t)=(x_1+x_o)^{-2m}\widetilde{\overline{u'_iu'_j}}(\tilde{x}_1)\ \text{and}$$
$$\ell_t(x_1,t)=(x_1+x_o)\tilde{\ell}_t(\tilde{x}_1), \tag{32}$$

where $\tilde{x}_1$ is taken from (31).

The corresponding dissipation function may immediately be taken from the small-scale equation (14) or even simpler directly from (32)

$$\varepsilon(x_1,t)=(x_1+x_o)^{-3m-1}\tilde{\varepsilon}(\tilde{x}_1), \tag{33}$$

where the relation

$$\varepsilon\sim\frac{K^{3/2}}{\ell_t} \tag{34}$$

has been invoked.

ℓ_t is linearly growing with x_1 independent of m. From experiments we usually have $m=0.43\dots0.75$ such that $\overline{u'_iu'_j}$ decreases algebraically with the distance from the turbulence source at $x_1=0$. $\tilde{x}_1$ is a typical diffusion type of similarity variable such as for the heat equation.

It is important to note that for the steady problem i.e. $t\to\infty$ we can show that all multi-point correlations such as R_{ij} become independent of $\tilde{x}_1$. Correspondingly the similarity variables for the one-point quantities such as $\widetilde{\overline{u'_iu'_j}}$, $\tilde{\ell}_t$ and $\tilde{\varepsilon}$ in (32) and (33) become constants. This may also directly be derived from (29) by omitting the part for the invariant surface corresponding to t and $\boldsymbol{r}$.

A sketch of the unsteady and the steady self-similar turbulent diffusion is given in figure 1.

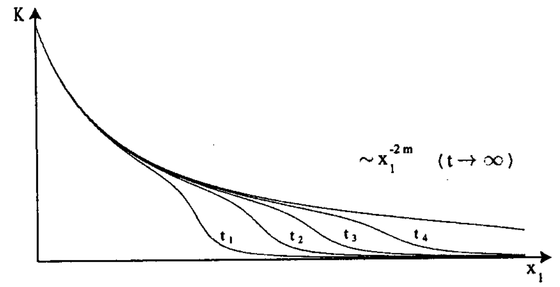

Figure 1: Sketch of the temporal evolution of the heat-equation-like turbulent diffusion process with linearly increasing integral length-scale according to (32).

Turbulent diffusion wave at a constant integral length-scale ($a_1 = 0$, $a_2 \neq 0$)

Since from the invariant surface condition (29) we can derive invariant solutions for arbitrary a_i's we may also impose certain symmetry breaking constraints. For the present case we impose $a_1 = 0$, which according to $x_1^* = e^{a_1} x_1$ in (23), corresponds to the symmetry breaking of scaling of space or in other words $a_1 = 0$ amounts to a constant integral length-scale.

Under this constraint and similar to (31) we obtain from (29)

$$\begin{aligned} \tilde{x}_1 &= x_1 - x_o \ln(t + t_o), \tilde{r} = r, \\ R_{ij}(x_1, t, r) &= e^{-2\frac{x_1}{x_o}} \tilde{R}_{ij}(\tilde{x}_1, \tilde{r}), \ldots . \end{aligned} \tag{35}$$

From (35) together with (34) we derive the corresponding one-point quantities

$$\begin{aligned} \overline{u_i' u_j'}(x_1, t) = e^{-2\frac{x_1}{x_o}} \widetilde{\overline{u_i' u_j'}}(\tilde{x}_1), \quad \ell_t(x_1, t) = \tilde{\ell}_t(\tilde{x}_1) \\ \text{and} \quad \varepsilon(x_1, t) = e^{-3\frac{x_1}{x_o}} \tilde{\varepsilon}(\tilde{x}_1), \end{aligned} \tag{36}$$

where the variable $\tilde{x}_1$ is taken from (35).

Equation (35) or rather (36) imply two important results. Due to the symmetry breaking of scaling of space, a diffusion-wave type solution is induced with decreasing amplitude in x_1-direction and decreasing wave speed proportional to $1/t$ as may be taken from $\tilde{x}_1$ in (35). Second, the spatial decay behavior in x_1-direction has changed from an algebraic to an exponential function.

The symmetry breaking of scaling of space or in other words the constant integral length-scale along the x_1-direction may be imposed by periodic boundary conditions in the x_2-x_3-plane such that in a direct numerical simulation the integral length-scale cannot grow.

Similar to subsection we may consider the corresponding steady case. The similarity variables of the multi-point correlations e.g. in (35) become independent of $\tilde{x}_1$. Similarly, $\widetilde{\overline{u_i' u_j'}}$, $\tilde{\ell}_t$ and $\tilde{\varepsilon}$ in (36) become constants. In particular the integral length-scale becomes a constant in space as $t \to \infty$.

A sketch of the diffusion wave and the corresponding long-time behavior is depicted in figure 2.

Turbulent diffusion in a constantly rotating frame ($a_1 \neq 0$, $a_2 = 0$)

In contrast to the previous case we may now consider the symmetry breaking of scaling of time in (24) due to $a_2 = 0$, imposed by an external time scale given by the frame rotation ($\tau = 1/|\mathbf{\Omega}|$). In the correlation equations (14), (14) or (17) frame rotation is modeled by invoking a non-zero $\mathbf{\Omega}$ or to be more specific in the present case $\Omega_1 \neq 0$.

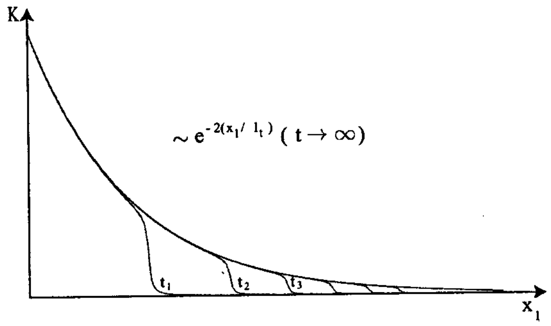

Figure 2: Sketch of the temporal evolution of the turbulent diffusion wave at constant integral length-scale according to (36).

In this case we find from (29)

$$\begin{aligned} \tilde{x}_1 = (x_1 + x_o) e^{-\frac{t}{t_o}}, \quad \tilde{r} = \frac{r}{x_1 + x_o}, \\ R_{ij}(x_1, t, r) = (x_1 + x_o)^2 \tilde{R}_{ij}(\tilde{x}_1, \tilde{r}), \ldots . \end{aligned} \tag{37}$$

The one-point quantities are similar to (32) and (33) with $a_2 = 0$ or rather $m = -1$ in (30) and may be written as

$$\begin{aligned} \overline{u_i' u_j'}(x_1, t) &= (x_1 + x_o)^2 \widetilde{\overline{u_i' u_j'}}(\tilde{x}_1), \\ \ell_t(x_1, t) &= (x_1 + x_o) \tilde{\ell}_t(\tilde{x}_1) \\ \text{and} \quad \varepsilon(x_1, t) &= (x_1 + x_o)^2 \tilde{\varepsilon}(\tilde{x}_1) \end{aligned} \tag{38}$$

where $\tilde{x}_1$ is defined in (37). Similar to the above, the steady case corresponds to the fact that the similarity variables in (37) and (38) become independent of $\tilde{x}_1$.

The surprising result for this case is that even for $t \to \infty$ the turbulent diffusion only influences a finite domain due to the quadratic behavior of the large-scale turbulence quantities.

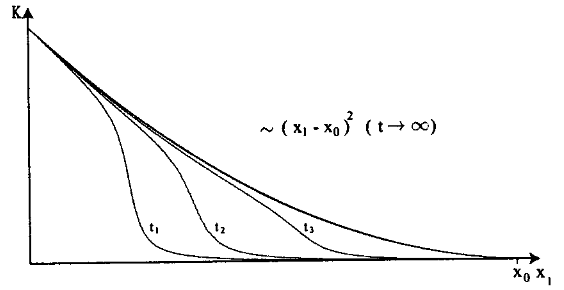

Figure 3: Sketch of the temporal evolution of the turbulent diffusion on a finite domain at a constant integral time-scale due to rotation according to (38).

MODEL IMPLICATIONS AND ANALYTIC MODEL SOLUTIONS

Classical two-equation models and Reynolds-stress transport models are investigated on its capability to capture one or several of the above three invariant solutions. The particular model equations we will investigate here are the classical K-ε (Jones & Launder, 1972) and the Launder-Reece-Rodi model (LRR) (Launder *et al.* , 1975). In addition, in the second subsection it will be shown that for the steady case a full analytic solution is given for the K-ε model. The invariant solutions are special cases of this full analytic solution.

Model implications derived from invariant solutions

Umlauf (2001) may have been the first who empirically recognized by numerically solving the steady diffusion problem employing the K-ε model that for certain model parameters a very distinguished change in model behavior appears. Without giving the invariant solution Umlauf derived the condition for the model parameter where the change in behavior occurred. We may suspect and this is what we see below that this change corresponds to a singular point in the model equation separating between an algebraic and an exponential decay with the singular point $m = -1$ or rather $a_1 = 0$. For each model this point corresponds to a certain set of model constants.

In order to test the compatibility of the invariant solutions derived in the last section with a certain turbulence model we simply implement the solutions into the model equations.

It is important to note that we only employ the steady solutions into the model equation, the rational behind this being the following. Formally, any of the invariant solutions (32)/(33), (36) or (38) for the unsteady problem lead to a reduction of the K-ε or the LRR model equations. However, these reduced set of equations may not have a useful set of solutions for physically appropriate boundary conditions. In particular the condition of boundedness for $x_1 \to \infty$ has to hold true for all time and hence also for $t \to \infty$ which corresponds to the steady solution. Hence implementing the steady solution into the model equations imposes a minimum requirement on the boundedness. In fact, one can show by asymptotic arguments that the condition of boundedness for the reduced equations of the unsteady case is the same as for the steady case. For this reason we only consider the steady case for the model investigation below.

If only one-dimensional diffusion is considered we find that for the generic case of no symmetry breaking the value of m which determines the spatial decay and the temporal behavior in (32)/ (33), is determined by a quadratic equation

$$\begin{aligned} &6(2\sigma_K - c_{\varepsilon_2}\sigma_\varepsilon)m^2 + 7\sigma_K m + \sigma_K = 0 \\ \Rightarrow \quad &m_{1,2} = \frac{7\sigma_K \pm \sqrt{\sigma_K(\sigma_K + 24c_{\varepsilon_2}\sigma_\varepsilon)}}{12(c_{\varepsilon_2}\sigma_\varepsilon - 2\sigma_K)} \end{aligned} \qquad (39)$$

and a quartic equation

$$\begin{aligned} &(456c_\varepsilon c_1 c_s c_{\varepsilon_2} - 144c_1^2c_\varepsilon^2 + 144c_1c_\varepsilon^2 \\ &\quad -336c_\varepsilon c_s c_{\varepsilon_2} - 216c_s^2c_{\varepsilon_2}^2)m^4 \\ &+(168c_1c_\varepsilon^2 - 168c_1^2c_\varepsilon^2 - 196c_\varepsilon c_s c_{\varepsilon_2} + 266c_\varepsilon c_1 c_s c_{\varepsilon_2})m^3 \\ &+(73c_1c_\varepsilon^2 - 28c_\varepsilon c_s c_{\varepsilon_2} - 73c_1^2c_\varepsilon^2 + 38c_\varepsilon c_1 c_s c_{\varepsilon_2})m^2 \\ &+(14c_1c_\varepsilon^2 - 14c_1^2c_\varepsilon^2)m - c_1^2c_\varepsilon^2 + c_1c_\varepsilon^2 = 0 \end{aligned} \qquad (40)$$

derived by implementing the invariant solutions into the classical K-ε and the Launder-Reece-Rodi model (LRR) (Launder *et al.* , 1975) respectively. From (39) and (40) and the standard model constants we respectively obtain the solutions for m: $m_1 = -0.14$, $m_2 = 2.49$ and $m_1 = -0.76$, $m_2 = -0.18$, $m_3 = -0.13$, $m_4 = 2.17$.

Since any of the values for m represent a solution of the corresponding model equations multiple algebraic solutions are admitted. This property of Reynolds averaged models is known to be important under certain conditions. In Durbin & Pettersson Reif (2001) it is shown that multiple solutions and the corresponding bifurcation of homogeneous shear flows are an important property which in fact models important turbulence physics.

The second case where the symmetry breaking of scaling of space is imposed, i.e. when a constant integral length-scale is considered, the solution (36) is only admitted if the model constants are modified. E.g. implementing (36) into the K-ε model, model constants need to obey the equation

$$\frac{c_{\varepsilon_2}\sigma_\varepsilon}{\sigma_K} = 2. \qquad (41)$$

This singular point is already visible in equation (39) since the denominator should not become zero for the exponent of the algebraic solution. The corresponding polynomial equations from the LRR model is given by

$$\begin{aligned} &(3c_1^2c_\varepsilon^2 - 12c_1c_\varepsilon^2 - 8c_1c_\varepsilon c_s c_{\varepsilon_2} + 112c_s c_\varepsilon c_{\varepsilon_2} - 144c_s^2c_{\varepsilon_2}^2)* \\ &(9c_1^2c_\varepsilon^2 - 12c_1c_\varepsilon^2 - 104c_1c_\varepsilon c_s c_{\varepsilon_2} + 112c_s c_\varepsilon c_{\varepsilon_2} + 144c_s^2c_{\varepsilon_2}^2) \\ &= 0. \end{aligned} \qquad (42)$$

Again we see at least for the LRR model that due to the two large factored terms different model parameters lead to multiple, here exponential, solutions.

It is important to note that for a given set of model constants only one solution type is admissible, either the algebraic solution (32)/(33) or the exponential solution (36). Note that the classical model constants do not solve the above equations (41) and (42). Hence, a one-dimensional solution in form of an exponential spatial decay is not admitted.

The steady version of the solutions (32)/(33) and (36) implemented into the model equations do not necessarily allow for independent boundary conditions for the Reynolds stresses and the dissipation. The steady form of the algebraic decay solution (32) and (33) implemented into the K-ε model leads to the relation

$$\left(\frac{\tilde{K}^3}{\tilde{\varepsilon}^2}\right)_{\text{alg}} = \frac{24\sigma_K(c_{\varepsilon_2}\sigma_\varepsilon - 2\sigma_K)^2}{C_\mu\left(7\sigma_K \pm \sqrt{\sigma_K(\sigma_K + 24c_{\varepsilon_2}\sigma_\varepsilon)}\right)^2}, \qquad (43)$$

where, evidently, only the positive sign has a physical meaning. Hence, once $\tilde{K}$ is picked $\tilde{\varepsilon}$ is determined. In contrast the steady version of the exponential decay (36) invoked into the K-ε model yields

$$\left(\frac{\tilde{K}^3}{\tilde{\varepsilon}^2}\right)_{\text{exp}} = \frac{x_0^2\sigma_K}{6C_\mu}, \qquad (44)$$

which, apparently, allows to freely chose $\tilde{K}$ and $\tilde{\varepsilon}$ due to the unconfined length scale x_0 which is proportional to ℓ_t.

Note that the dimensions of $\tilde{K}$ and $\tilde{\varepsilon}$ are different for the two cases above. According to (32)/(33) $\tilde{K}$ and $\tilde{\varepsilon}$ have fractional dimensions in (43) while from (36) we find that in (44) $\tilde{K}$ and $\tilde{\varepsilon}$ have the same dimensions as the original variables K and ε.

Interesting enough the singular point (41) is also visible in the boundary relation (43). At this point the nominator becomes zero. In addition also the denominator vanishes if the minus sign is chosen.

The third case of a rotating frame, here rotating about x_1, cannot be modelled at all by one-point models. Classical linear two-equation models are insensitive to rotation. However, even fully non-linear Reynolds-stress transport models (non-linear in the Reynolds-stresses) are insensitive to rotation about x_1 for the present flow, elucidating a serious shortcoming of these models. There appears to be only one model which may account for system rotation in the present pure diffusion case. It is the model by Sjögren & Johansson (2000) which is non-linear in the mean-velocity gradient.

SUMMARY AND CONCLUSIONS

A set of three different invariant solutions for the turbulent diffusion problem have been constructed based on Lie group analysis of the multi-point correlation equations. The solutions cover classical diffusion-like solution (heat equation like) with algebraic spatial decay, decelerating diffusion-wave solution with exponential spatial decay and finite domain diffusion due to rotation. Two-equation model equations and full Reynolds stress equations have been investigated whether they capture any of the invariant solutions. Particularly the classical K-ε and the LRR model have been investigated. All models comply with the diffusion-like solution with algebraic spatial decay. The decay exponent is determined by the model constants while multiple decay exponents are observed. The exponential solution is only admitted by the model equations if model constants obey certain algebraic relations. For a given set of model constants either the algebraic or the exponential solution is admitted. None of the classical models is sensitive to rotation for the present diffusion problem and hence the last solution of diffusion on a finite domain is not admitted by any of the turbulence models. The only exception might be the model by Sjögren & Johansson (2000).

We should note that the discrepance between the admitted invariant solutions for the multi-point correlation equations and the one-point model equations lies in the reduced dimensionality of the one-point equations. For the set of classical model constants only the algebraic solution is obtained. Nevertheless, we may not conclude from the one-dimensional case that with the classical model constants exponential solutions are not admitted for the two- or three-dimensional case. In fact, it appears to be very likely that these solutions exist for probably all Reynolds stress models at dimensions higher than one.

The case of turbulent diffusion with rotation is very difficult. It may only be modelled with a new model equation which is fully non-linear in the mean-velocity gradient. A new model development appears to be necessary and may be along the lines of the model by Sjögren & Johansson (2000).

The latter two questions are the topic of present research and will be published elsewhere.

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