CASCADE BEHAVIORS OF VORTICES INTERACTIONS

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ABSTRACT

The dynamic behaviors of four vortex rings ' interaction are analyzed from several new viewpoints. A correlation function of rotation with deformation of a fluid particle, a measure of the nonlinear interaction between rotation and deformation in average of physical space, and the Kolmogorov entropy in rotation-deformation space are introduced to describe the behaviors of complex flows. The first two undergo nearly opposite experience in the cascade process and have steady asymptotic states. The entropy increases within the cascade process, which is consistent with the disordered nature of turbulence. To reveal the characteristics of the vector, $W_i = \omega_j S_{ji}$, which appears in the vorticity transport equation and whose role is to stretch and distort vorticity, a two-dimensional phase space is proposed, in which the relative orientations of the vector against vorticity vector are focused. It is found that the distribution of the vector in the phase space is limited in a well-bounded domain. The events of vorticity stretching are shown more than those of compressing and this is attributed to the nonnegative source term in its governing equation. While for the concentrated vorticity, the alignment of the vorticity stretching and distorting vector $\mathbf{W}$ with vorticity vector is obvious, the statistic alignment between them does not observed.

INTRODUCTION

Turbulence can be regarded as flow fields with various scales of vortices coexisting when Reynolds numbers of the flows are high enough. Interactions of these vortices dominate the dynamical behaviors of the flows such as energy cascades. Although the number of vortices in a real turbulence is very huge and their interactions are quite complicated, it can be expected that studying interactions of two or more vortices may lead to revealing the fundamental features of turbulent dynamics. The interactions of vortices themselves are also of theoretical interest in fluid mechanics. For these reasons, the behaviors of two vortex tubes or vortex rings have been widely studied (e.g. Moffatt, 2000, Marshall et al., 2001, Allen and Auvity, 2002, and references therein).

Thanks to the advancement of computer power, it is now possible to obtain a database of three-dimensional complex flows in detail from direct numerical simulations. Compared to the velocity field, the field of velocity gradient, consisting of deformation tensor and rotation tensor, may disclose more mechanisms and features of fluid flows. Mathematically, frame-independent scalars of a tensor are preferred to depict the properties of the tensor. The two well-known and physical meaningful invariants are the enstrophy $\omega^2 = \omega_i\omega_i$ and the deformation modulus or strain strength $S^2 = S_{ij}S_{ij}$. Here, ω_i is the vorticity vector and S_{ij} is the deformation tensor. The spatial characteristics of these two invariants have been extensively documented. However, in the past decade, the investigation of other invariants besides the two were done more and more. Specifically, the orientation, in statistics, of vorticity vector in the frame of the principal axes of the deformation tensor has attracted more attentions (e.g. Andreotti, 1997 and references therein). It has been discussed in the cases of homogeneous isotropic turbulence (Nomura and Post, 1998, Tsinober, 1998), homogeneous sheared turbulence (Nomura and Diamessis, 2000), and atmospheric surface layer (Kholmyansky et al., 2001). It was shown that there exists the aligning tendency between the vorticity vector and the intermediate eigenvector of the deformation tensor. The similar tendency between the vorticity and the vector making vorticity tube stretched and distorted, $W_i = \omega_j S_{ji}$, which appears in the transport equation of vorticity, was also shown by Tsinober et al. (1992). It should be pointed out here that there is a strong correlation between the two alignments because the alignment of vorticity vector with $\mathbf{w}$ means vorticity vector is one of the eigenvectors of the deformation tensor. Moreover, Tsinober et al. (1999) discussed the behaviors of the key nonlinearities related to the velocity gradients in flow regions dominated by enstrophy and strain. And very recently, Chong et al. (2002) overviewed a method of using the invariants of the velocity gradient tensor to study eddying motions and turbulences. Their approach is of topological methodology by extending the critical point theory.

In this paper, a DNS database of four ring-vortices with identical intensity colliding to each other in a cubic box is used to analyze the dynamic behaviors of the complicated multi-vortices interaction. The simulation was carried out by a spectral method where periodic boundary condition was applied. The Reynolds number of the flow, based on the circulation of a vortex ring, is 300. Energy cascade can be clearly seen in spectral space before they dissipate, although the Reynolds number of the computation is not sufficiently high. The analyses are based on some various phase spaces and variables, which consist of the invariants of the velocity gradient field and, we believe, pose more important physical meaning and are more appropriate for non-periodic boundary flows. By looking into the behaviors of the flow field in these phase spaces and variables, some new fundamental features of cascade behaviors are revealed.

DYNAMICS OF THE INVARIANTS OF VELOCITY GRADIENT TENSOR

It has been known that the number of the *independent* invariants of *any* asymmetrical tensor is not more than six in the sense that each extra-invariant has an algebraic constraint (linear or nonlinear) equation relating it to the six independent invariants. By decomposing the asymmetrical velocity gradient tensor $u_{i,j}$ into symmetric part $S_{ij} = (u_{i,j} + u_{i,j})/2$ and antisymmetric part $R_{ij} = (u_{i,j} - u_{i,j})/2 = -\epsilon_{ijk}\omega_k/2$, besides the trivial invariant $S_{ii} \equiv 0$ for incompressible flows, the other five concise independent invariants may be chosen as:

$$J_1 = S_{ij}S_{ij} \tag{1}$$

$$J_2 = S_{ij}S_{jk}S_{ki} \tag{2}$$

$$I_1 = \omega_i\omega_i \tag{3}$$

$$I_2 = \omega_i S_{ij}\omega_j \tag{4}$$

$$I_3 = \omega_i S_{ij} S_{jk}\omega_k \tag{5}$$

Therefore, all we need to know in this context are the properties of the three physically meaningful variables themselves, i.e. strain, vorticity, and **W** vector, and the relative orientation of the **W** vector with vorticity vector. Although these five quantities are free from each other in the tensor theory viewpoint, they are closely coupled by a dynamic system which can be derived from the incompressible Navier-Stokes equations:

$$\frac{DJ_1}{Dt} = -2J_2 - \frac{1}{2}I_2 - 2S_{ij}\frac{\partial^2 p}{\partial x_j \partial x_i} - 2\nu\frac{\partial S_{ij}}{\partial x_l}\frac{\partial S_{ji}}{\partial x_l} + \nu\frac{\partial^2 J_1}{\partial x_l \partial x_l} \tag{6}$$

$$\frac{DJ_2}{Dt} = -\frac{3}{2}J_1J_1 + \frac{3}{4}J_1I_1 - 3S_{ij}S_{jk}\frac{\partial^2 p}{\partial x_k \partial x_i} - 6\nu S_{ij}\frac{\partial S_{jk}}{\partial x_l}\frac{\partial S_{ki}}{\partial x_l} + \nu\frac{\partial^2 J_2}{\partial x_l \partial x_l} \tag{7}$$

$$\frac{DI_1}{Dt} = I_2 - 2\nu\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} + \nu\frac{\partial^2 I_1}{\partial x_l \partial x_l} \tag{8}$$

$$\frac{DI_2}{Dt} = I_3 - \omega_i\frac{\partial^2 p}{\partial x_i \partial x_j}\omega_j + \nu\frac{\partial^2 I_2}{\partial x_l \partial x_l} - 2\nu\left(\frac{\partial \omega_i}{\partial x_l}\frac{\partial S_{ij}}{\partial x_l}\omega_j + \frac{\partial \omega_i S_{ij}}{\partial x_l}\frac{\partial \omega_i}{\partial x_l}\right) \tag{9}$$

$$\frac{DI_3}{Dt} = -2\omega_i\frac{\partial^2 p}{\partial x_i \partial x_j}S_{jk}\omega_k + \nu\frac{\partial^2 I_3}{\partial x_l \partial x_l} - 2\nu\left(2\frac{\partial \omega_i}{\partial x_l}\frac{\partial S_{ij}}{\partial x_l}S_{jk}\omega_k + \frac{\partial \omega_i S_{ij}}{\partial x_l}\frac{\partial S_{jk}\omega_k}{\partial x_l}\right) \tag{10}$$

Here, $D(*)/Dt$ stands for the material derivatives. Note that for two-dimensional flows, $J_2 = I_2 = I_3 = 0$, indicating only global effects exist for all variables, and that I_3 serves as production term of I_2 and the latter plays the same role on I_1. Because of the non-negative characteristics of I_3, the predominance can be expected of vorticity stretching over compressing.

As shown in Eqns.(6), (7), (9), and (10), the pressure Hessian plays an important role in the development of the system. While the effects of the anisotropic part of the Hessian are difficult to analyze due to its global feature, the influence of the isotropic part of the Hessian:

$$\frac{\partial^2 p}{\partial x_l \partial x_l} = \frac{1}{2}I_1 - J_1 \tag{11}$$

is local and is explicit in the governing equations of the system. Eqn.(11) is a Poisson equation from which the pressure can be solved and then the pressure Hessian can be derived. Here, the combination of enstrophy and deformation strength in the form of the left hand of Eqn.(11) serves as the 'source of pressure'.

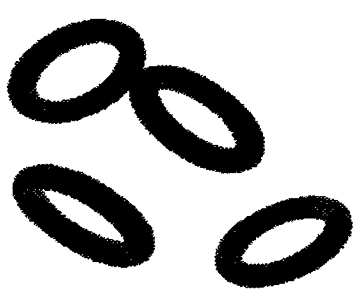

Fig.1 Impinging of four vortex rings

SIMULATIONAL RESULTS AND DESCUSSIONS

Fig.1 shows the iso-surface of enstrophy of the four vortex tubes in the initial state of the simulation. In the time evolution that follows, they interact each other and induce complicated phenomena. The evolution can be divided into three scenarios in the view of the development of the energy spectra. In the first scenario (from timestep1 to timestep5), no obvious energy transfer can be observed as shown in Fig.2(a). In the second (from timestep6 to timestep15), shown in Fig.2(b), the energy of the smaller scales increase remarkably while that of the larger scales decrease, note that the total energy decay almost exponentially throughout the entire process as shown in Fig.2(d). In the final scenario, the energy of all of scales decay quickly as shown in Fig.2(c). Because the second scenario shows clear cascade behavior, it will be focused in this paper. The evolution of the total enstrophy is also shown in Fig.2(d). From the beginning of the cascade the enstrophy starts growing up. But the moment it begins to decay is a lot ahead of the end of the energy cascade (gray region in the diagram). Hence the growing process of enstrophy cannot be used to identify the cascade process. A cascade tracer that is independent from spectrum is desirable for non-periodic boundary condition flows.

From the kinematic viewpoint, the deformation and the rotation of fluid particles are two independent attributions of fluid media. In other words, the fluid flow can be either of potential flow or of purely spinning motion without any deformation. But the cases in which they are dynamically coupled are more common and more important in the theoretical and engineering viewpoints. Fig.3 shows the distribution of the PDFs in rotation-deformation space, respectively of the initial state and of the late state of cascade. It can be observed that the distribution evolves from an organized state to a scattered state, the pattern of the latter is similar to the case of homogeneous sheared turbulence (Tanaka and Kida, 1993). It should be noted here that there exists a constraint that the source and sink of pressure must balance with each other due to periodic boundary condition. Indeed, the Kolmogorov entropy increases during cascade process (see gray region in Fig.4). Nevertheless, the correlation between enstrophy and deformation is not reduced as a result of disorder but enhanced as shown in Fig.4. in the figure the development of the special-average correlation function of I_1 with J_1 is shown. The development of

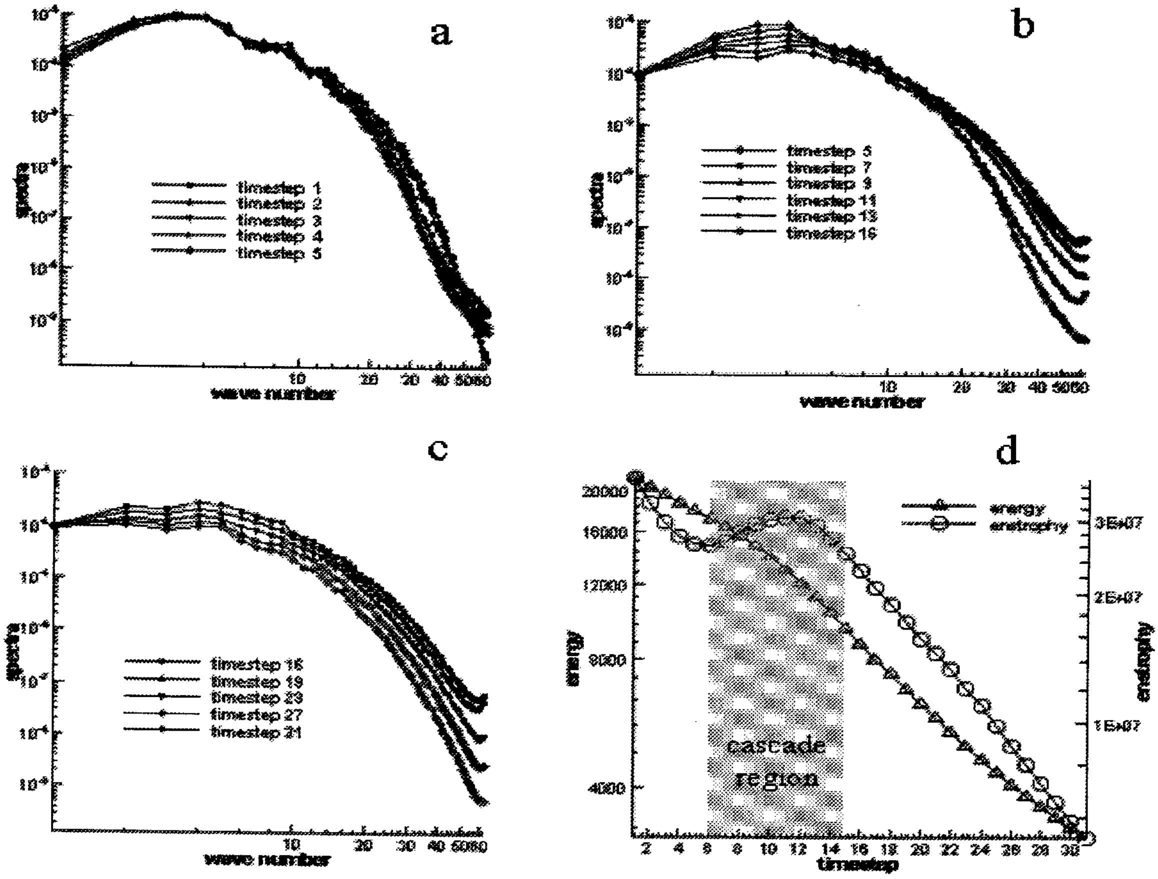

Fig.2 Development of energy spectra (a,b,c) and total energy and enstrophy (d)

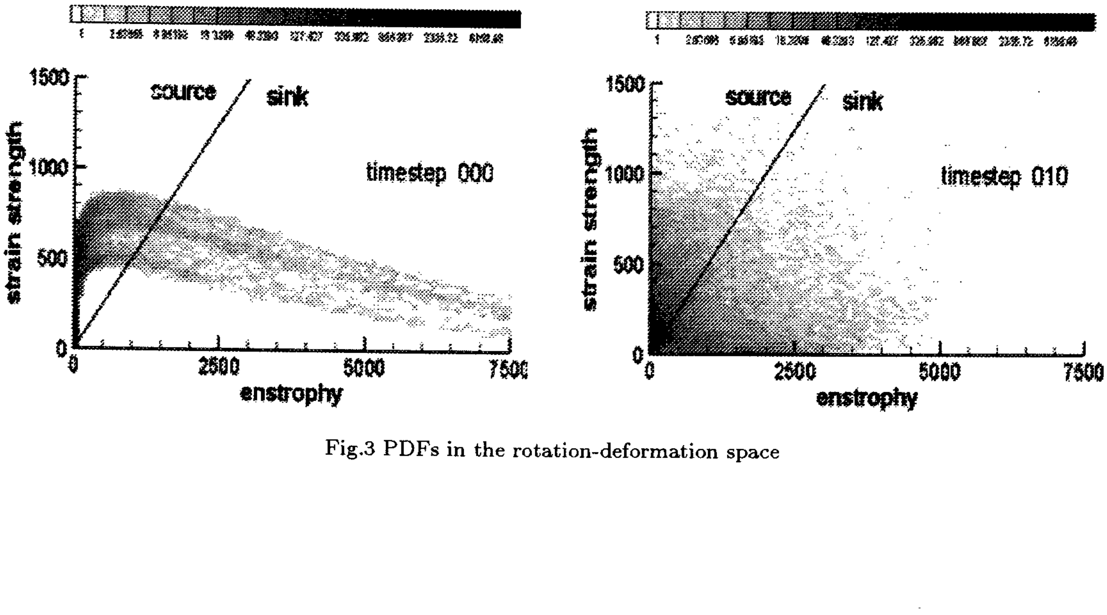

Fig.3 PDFs in the rotation-deformation space

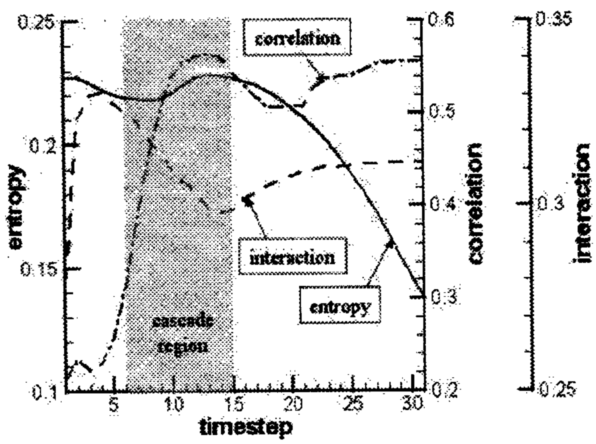

Fig.4 Development of variables

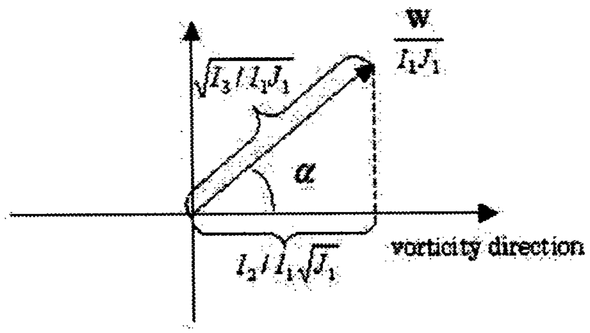

Fig.5 Diagram of the relative orientation of **W** vector with vorticity vector

the normalized production term $< I_3/I_1J_1 >$ is also plotted in Fig.4. Here $< * >$ denotes the special average. It is worthy to point out that $< I_3/I_1J_1 >$ may also be regarded as the nonlinear interaction between rotation and deformation. The behavior of this variable in the figure suggests the cascade be also a relax process of nonlinear interaction between deformation and rotation, though their correlation increases. The cascade probably can be more appropriately defined as excitement of the entropy, the range of which is a little narrower than the origin but the transitional moment of end of process is in better agreement with that of the other two variables. Because the correlation and the interaction term exhibit steady asymptotic behaviors in our case, another question of interest, which is beyond the limit of this paper, is if there are such asymptotic values of them that they are only dependent on Reynolds number in fully developed turbulence.

Based on the vorticity transport equation, it can be assumed that development of the vorticity be mainly affected by the **W** vector. This vector plays one of two important roles in energy cascades of the three-dimensional turbulences. To reveal its characteristics and to understand its dynamic behaviors, a two-dimensional phase space is designed as shown in Fig.5. The abscissa in Fig.5 is parallel to the direction of local vorticity vector of flow field and a point in this space, determined completely by a set of invariants, orientates the normalized **W** vector relative to the local vorticity: the distance from the origin of the coordinates system identifies the magnitude of the vector and the α suggests the angle between the vector and the local vorticity. It should be stressed here that the relative orientation of the **W** vector with the local vorticity is important rather than the absolute direction of the vector in the physical space. Fig.6 separately shows the PDFs of the identical relative orientation of the two vectors in the initial state and in the late state of cascade of our case. The first of interest is that the distribution is bounded in a zone whose boundary seems to be able to be described as the upper half of an ellipse. But what can be demonstrate now is just that $|I_2/I_1\sqrt{J_1}|$, i.e. the half of the longer axis of the ellipse if it is, cannot be over $\sqrt{2/3}$ for incompressible flows. The half of the shorter axis is suggested to be $\sqrt{1/2}$ based on more analyses. We believe that this boundary is just a consequence of the normalization and is a merit to do analyses. Another distinct observation in the figures is that the events with the tendency of alignment between the **W** vector and the vorticity vector are very rare. This is quite different from the cases referred in the introduction section. On the other hand, the concentrated vortex tubes are of rare events. Fig.7 exhibits the distributions of the enstrophy averaged among those that share same point in the space. It can be seen that intense vorticity is located in the zone of the rare events with tendency of alignment. They develop from symmetric pattern to positively-skewed position which suggests that the vortex rings are elongated during the cascade, just as expected in the last section. This confirms that the vorticity stretching is firmly related to the energy cascade. It can be also justified from our results that the major contribution to the normalized nonliear interaction come from neither the intense vorticity structures nor the strong strain structures (see Fig.7 and Fig.8). Fig.8 shows the distributions of the averaged strain strength. The distribution is similar to that of the averaged enstrophy in the late stage of cascade, which is in consistent with the high correlation shown in Fig.4.

CONCLUSIONS

The analyses of DNS results have shown that the energy cascade process can be exhibited in a system wherein only four vortex rings are interacting with each other. This confirms the importance of interaction between vortices in turbulent cascade. The inspection is firstly performed for the behaviors of the correlation of rotation with deformation of fluid particles, the normalized nonlear interaction between rotation and deformation in average of the physical space, and the entropy in rotation-deformation space. The aim is to find out an alternative method of identification of cascade, which is applicable for non-periodic boundary flows. All of them are found undergoing uneven experience in the cascade regime. The behaviors of the entropy is proposed to be used to identify the cascade process. However more cases need to be analyzed for this purpose. It is also found that, in the vortex interaction, the balance of "source of pressure" changes from an ordered state to a disordered state in the rotation-deformation space with increase of entropy.

To reveal the characteristics of the normalized vorticity stretching and distorting vector, a two-dimensional phase space is proposed, in which the relative orientations of the vector with vorticity are focused. It can be clearly shown in this space that the stretching is predominant rather than the compressing in the flow, especially for concentrated vorticity. The reason is attributed to the non-negative source term in its governing equation. Nevertheless, unlike the cases of other researchers, no tendency, in statistics, of alignment between the vector and vorticity is evolved, though it does for

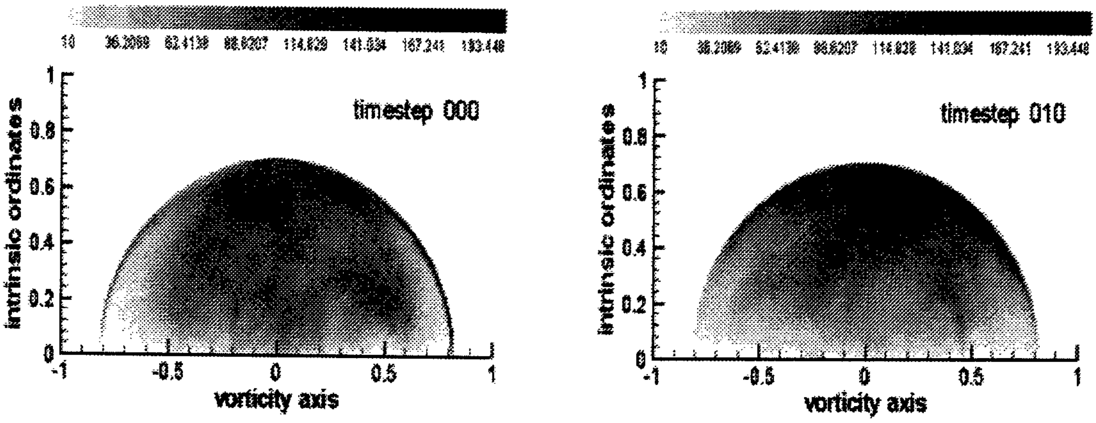

Fig.6 PDFs of the normalized vorticity stretching and distorting vector $\mathbf{W}/I_1 J_1$

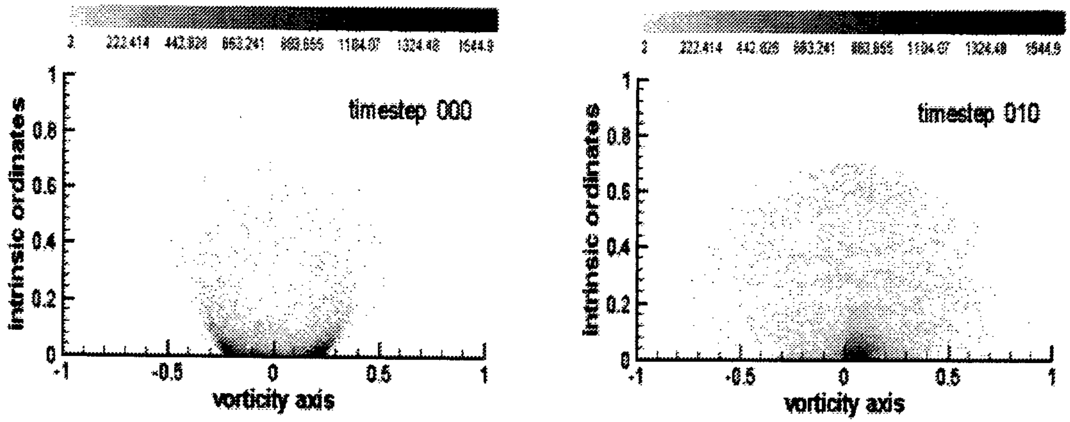

Fig.7 Distributions of the averaged enstrophy

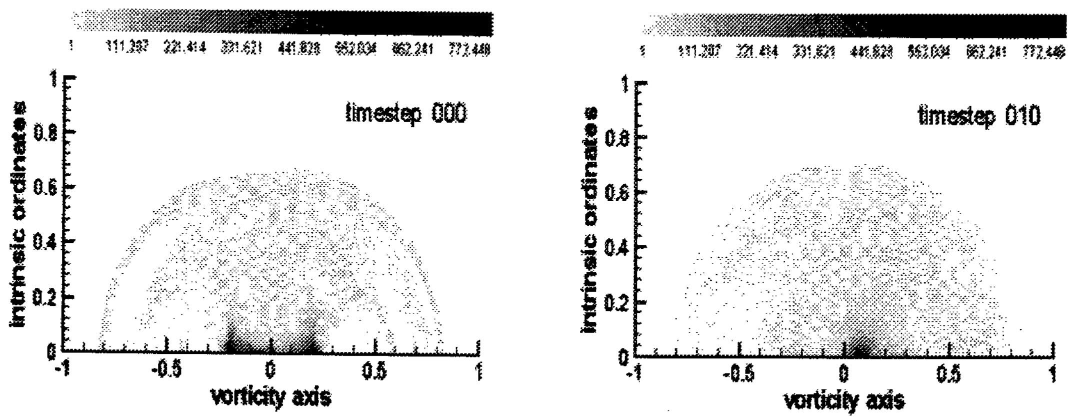

Fig.8 Distributions of the averaged strain strength

concentrated vorticity structures. The latter's contributions to the normalized interaction are found quite small.

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