INVARIANT ASSUMPTION OF PDF AND MEAN VELOCITY PROFILE IN HIGH-REYNOLDS NUMBER TURBULENT BOUNDARY LAYERS

Yoshiyuki Tsuji
Department of Energy Engineering and Science,
Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan
c42406a@nucc.cc.nagoya-u.ac.jp

Björn Lindgren, and Arne V. Johansson
Department of Mechanics,
Royal Institute of Technology (KTH)
SE-100, 44 Stockholm, Sweden

ABSTRACT

The probability density function (pdf) of a stream-wise velocity component is studied in zero-pressure gradient boundary layers. From analyzing the data up to $R_\theta \simeq 13000$, it was found that pdfs have self-similar profiles ranging from $y^+ \simeq 180$ to $0.049 \cdot \Delta^+$, where Δ is Rotta-Clauser boundary layer thickness. Pdf profiles asymptote to the universal shape very close to the Gaussian, but are positively skewed at the core region, indicating smaller values in the tail parts. Based on this experimental fact, the mean velocity profile is reconsidered from the standpoint of pdf equation. The log-law profile is expected as the mean velocity distribution.

INTRODUCTION

The probability density function of stream-wise velocity component is studied in the zero-pressure gradient boundary layers. The focus is attracted to investigate the self-similarity of probability density functions (abbreviated as pdf hereafter) in the over lap region. This study was motivated by the report (Tsuji & Nakamura, 1999), in which pdf's profile was analyzed and the mean velocity distribution was discussed. A logarithmic velocity profile was derived from pdf equation subject to the ***invariant assumption of pdf***, which means that pdf shape becomes self-similar in the over-lap region, and a few empirical relations.

In the TSFP-1 (1999 , Santa barbara) the basic idea was introduced and its efficiency was confirmed in low-Reynolds number turbulent boundary layers (Tsuji et al., 1999). The idea was extended in the case of rough-wall boundary layers in TSFP-2 (2001 , Stockholm). The over lap region in smooth and rough wall boundary layers were compared with each other from the view point of pdf's profiles (Tsuji et al., 2001). In the course of above researches, as the next step, we are interested in pdfs in high-Reynolds number flow. So, the first purpose in this paper is to check the invariant assumption in high-Reynolds number turbulent boundary layers ($3000 < R_\theta < 13000$) measured in MTL wind-tunnel at KTH. Second, we remark the recent experiments regarding the mean velocity profiles in high-Reynolds number flow (Österlund, 1999; Österlund et al., 2000). We suggest that the log-law profile is appropriate as a universal velocity distribution in the over-lap region within the framework of pdf equation.

KULLBACK-LEIBLER DIVERGENCE

We will review briefly the Kullback-Leibler divergence (abbreviated as K.L. divergence hereafter) for convenience. Its detailed explanation is described in the reference (Amari, 1985). The statistical model $S = \{p(x, \xi)\}$ parametrized by ξ is defined, where x is a random variable belonging to a sample space X, and $p(x,\xi)$ is the probability density function. Here, ξ is a real n-dimensional parameter $\xi = (\xi^1, \xi^2, \cdots, \xi^n)$ belonging to some open subset Ξ of the n-dimensional real space R^n. When $p(x,\xi)$ is sufficiently smooth in ξ, it is natural to introduce in a statistical model S the structure of an n-dimensional manifold, where ξ plays the role of a coordinate system. When the metric tensor is introduced for each ξ, Riemanian space is constructed. The K.L. divergence is a generalization of the Riemanian metric in space S, which is the most important quantity about the differential geometry in statistics.

The meaning of the K.L. divergence is explained concretely as follows. Let us assume that a discrete probability distribution $\{p_i^0 : i = 1, 2, \cdots, n\}$ with $p_i^0 \neq 0$ over a sample set is given. If this probability changed to $\{p_i : i - 1, 2, \cdots, n\}$ by a new information,

$$b_i^0 - b_i = \ln(p_i/p_i^0)\,, \tag{1}$$

is defined as the information change. As $b_i\,(= -\ln p_i)$ is the the knowledge with respect to the event i, the decrease of p_i describes the increase of the knowledge. On average, the statistical weight of the result i is determined by the new probability p_i, and we obtain the mean value of Eq. (1) as,

$$K(p, p^0) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{p_i^0}\,, \tag{2}$$

which is the definition of the K.L. divergence and is also called the *information gain*. $K(p,p^0) \geq 0$ is zero for only $p = p_0$,therefore, it is considered to indicate the difference between $\{p_i\}$ and $\{p_i^0\}$.

PROBABILITY SHAPE IN OVERLAP REGION

Kullback Leibler divergence (KLD) was used to evaluate the shape of pdf qualitatively. This measure is defined in this data analysis as follows,

$$D(P||Q) \equiv \sum_{\{s\}} P(s_i) \log_e \left(P(s_i)/Q(s_i)\right)\,, \tag{3}$$

where $P(\mathbf{s})$ and $Q(\mathbf{s})$, $\{\mathbf{s}\} = \{s_1, s_2, \cdots\}$, are discrete probability distributions. KLD has a non-negative value for any $P(\mathbf{s})$ and $Q(\mathbf{s})$, and it is zero only when $P(\mathbf{s})$ is exactly equal to $Q(\mathbf{s})$. In this analysis, $Q(\mathbf{s})$ is set to be Gaussian profile and $P(\mathbf{s})$ is set to the pdf of velocity fluctuation, $P_y(s)$, measured at location y from the wall. The typical example of mean velocity and KLD are plotted in Fig. 1. The solid symbols indicate the log-region suggested by Österlund (1999). The solid line is the log-law profile, $U^+ = (1/\kappa) \cdot \log_e y^+ + B$ with $\kappa = 0.38$ and $B = 4.1$. It is clear that there is a constant KL-divergence region. This means that P_y does not change. Its beginning point is expressed as y_s^+ and its end is y_e^+, respectively.

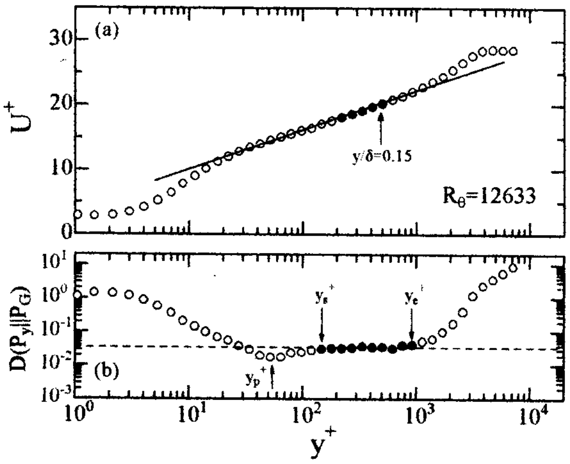

Figure 1: Typical example of mean velocity profile and KL-divergence distribution. The solid circles indicate the log-law region suggested by Österlund et al. (1999): $200 \le y^+ \le 0.15 \cdot \delta^+$. KL-divergence is defined by Eq. (3) where P_y indicates the probability at location y, and P_G is Gaussian profile. Solid circles in the bottom graph indicate the constant KL-divergence region. The beginning and end point are expressed as y_s^+ and y_e^+, respectively. y_p^+ is the point of mimum KL-divergenve.

Both y_s^+ and y_e^+ are plotted in Fig. 2. The former is $y_s^+ \simeq 180$ but the latter increases as a function of Reynolds number based on momentum thickness, which is located outside of $y = 0.15 \cdot \delta$. Analyzing the KTH database, it was found that pdfs have self-similar profiles in the extent from $y^+ \simeq 200$ to $0.3 \cdot \delta$, where δ is a boundary layer thickness. Pdf profiles asymptote to the universal shape very close to the Gaussian, but are positively skewed at the core region, indicating smaller values in the tail parts (Lindgren et al., 2002).

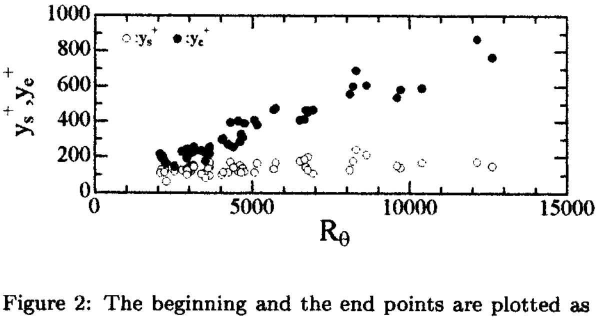

Figure 2: The beginning and the end points are plotted as a function of Reynolds number; $R_\theta \equiv U_0\theta/\nu$.

MEAN VELOCITY PROFILE

The detailed explanation is omitted here (Tsuji & Nakamura, 1999), however the following rational expansion was used to interpolate the turbulence intensity distribution in the inner region.

$$u_r^+ = \alpha + \beta_1 (y^+ - \gamma)^{-1} + \beta_2 (y^+ - \gamma)^{-2} + \cdots . \quad (4)$$

The coefficient α means $\alpha = \lim_{y^+ \to \infty} u_r^+$ and β_i represent the decay of u_r. γ is an open parameter but is a small quantity. In this analysis γ is set at 5, and the rational expansion Eq. (4) is valid from $y^+ \simeq 20$ to y_r^+. The outer end y_r^+ is empirically evaluated as $0.15 \cdot \delta^+ \le y_r^+ \le 0.2 \cdot \delta^+$. We have derived the log-law profile from pdf equation subject to Eq. (4) (Tsuji & Nakamura, 1999).

$$U^+ = C_1 \left\{ \alpha\beta_1 \log_{10}(y^+ - \gamma) + \frac{2\alpha\beta_2 + \beta_1^2}{y^+ - \gamma} + \cdots \right\} + B . \quad (5)$$

Sufficiently far from the wall, the leading term is dominant, and Eq. (5) represents the logarithmic profile. At close to the wall, the contribution from the second term should be considered. The coefficient C_1 is obtained by the information of $-\langle uv \rangle / u_\tau^2$, and it is set at 0.2 in this analysis. α is scaled like $\alpha = 0.33 \cdot R_\theta^{0.21}$ and $\beta_1 = 91.8 \cdot R_\theta^{-0.21}$. Therefore, within the experimental accuracy, the product of α and β_1 is constant independent of the Reynolds number, that is, $\alpha\beta_1 \simeq 30.3$. The slope of logarithmic profile is computed as $C_1\alpha\beta_1 = 6.06$, or Kármán constant, κ, is rewritten as $\log_e 10/(C_1\alpha\beta_1)$, thus we have $\kappa = 0.379$. On the additive constant B, the theoretical procedure ascertains that it depends on the Reynolds number but approaches the asymptotic value.

Logarithmic profile is expected in the intersection of above two regions; $[y_s^+, y_e^+] \cap [20, y_r^+] = [y_s^+, y_r^+]$. Using the least square approximation, the constant κ and B are uniquely determined within $[y_s^+, y_r^+]$. In Fig. 3 they are plotted as a function of Reynolds number. We have $\kappa \simeq 0.38$ and $B \simeq 4.1$ for $5000 \le R_\theta$. These results are consistent with ones obtained by Österlund et al. (2000), who analyzed carefully the mean velocity gradient, but here we thought of this problem in a different way. It is noticed two different approaches indicate the same results.

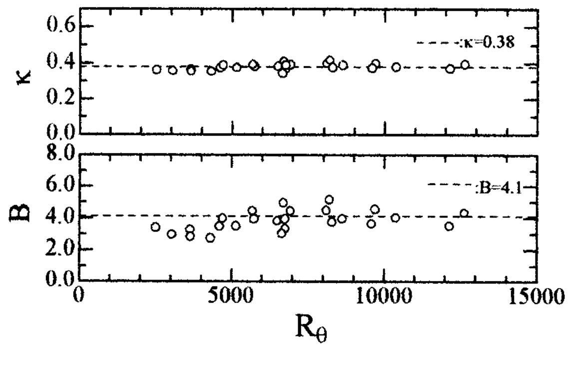

Figure 3: Coefficient κ and B are obtained by fitting $U^+ = (1/\kappa) \ln y^+ + B$ to the experimental data within $[y_s^+, y_r^+]$.

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