UNIVERSALITY OF TEMPERATURE STATISTICS IN STABLY STRATIFIED TURBULENCE

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ABSTRACT

Using two dimensional direct numerical simulations, the statistics of the temperature fluctuations in stably stratified turbulence are studied. Comparison with passive scalar statistics in similar flows suggest that the stably stratified case is qualitatively similar to the passive case. A stratification-dependent mixing length is shown to be the dominant factor in determining rare event temperature statistics. Probability density functions at various stratification strengths collapse to the same functional form as in the passive case when rescaled using this length scale.

INTRODUCTION

The mixing of a passive scalar field in turbulent flows has been studied in great detail (Warhaft, 2000). Recent work has suggested that large-scale anisotropies in the flow persist down to the smallest scales, and this has indicated a new kind of universality in the scalar statistics of rare events (Celani, *et. al.*, 2000). It is of great interest to understand if these ideas extend to the case of an active scalar field. Stably stratified turbulence is a prime example of an active scalar field, with the scalar coupled to the fluid flow via buoyancy. It has relevance to many geophysical and astrophysical fluids. In these flows, a stable density gradient is imposed on a fluid, which is then stirred to generate turbulence. The density gradient is often the result of an applied temperature gradient, since in most fluids density is proportional to the temperature. The presence of a gravitational acceleration along the direction of this applied gradient dynamically couples the temperature field to the momentum equation of the fluid. In the absence of gravity this scenario reduces to the passive scalar case, while increasing the magnitude of the gravitational acceleration strengthens the coupling to the dynamics. This coupling influences the large scale motions, determining a mixing length which apparently controls a variety of rare event statistics.

Using two-dimensional numerical simulations, we find evidence that the universality of the passive scalar rare event statistics found by Celani, *et. al.*, (2000) extends to these stably stratified flows. In the passive case, the probability density functions (PDFs) of the rare event statistics of temperature differences at various separations collapse when rescaled by a power of the separation. We find that this also occurs in the stably stratified case. In addition, we find that a single stratification-dependent length scale can be used to collapse the rare event statistics at different stratification strengths as well as different separations. This same length scale also permits the collapse of the PDFs of the temperature values and temperature gradients.

Problem Description

To examine the statistics of the temperature field in stably stratified turbulence, we utilize two dimensional simulations analogous to those employed in Celani, *et. al.* (2000), and Boffetta, *et. al.* (2000), for the passive scalar case. Following the procedures for the numerical simulation of the two-dimensional incompressible Navier-Stokes equations described in

Boffetta, *et. al.* (2000), we define the velocity field using the stream function $\psi(\mathbf{x},t)$, from which the horizontal velocity component $u = \partial_z\psi$ and the vertical component $w = -\partial_x\psi$ are derived. In addition, we have a temperature field $T(\mathbf{x},t)$ which evolves with the flow and couples to the fluid density via the buoyancy term in the momentum equation. In the Boussinesq limit, the two-dimensional Navier-Stokes equations are expressed in terms of the vorticity $\omega(\mathbf{x},t) = \nabla^2\psi$ and ψ as

$$\partial_t\omega + J = \nu_n\nabla^{2n}\omega - \lambda\omega - \nabla^2 f - g\alpha\partial_x T, \quad (1)$$

where $J \equiv \partial_z\psi\partial_x\omega - \partial_x\psi\partial_z\omega$ is the usual Jacobian, f is a forcing function which adds energy to the flow at small scales, $\lambda\omega$ is a friction term which dissipates energy at large scales, g is the gravitational acceleration, and α is the volume expansion coefficient of the fluid. The temperature is passive if the coupling $g\alpha$ is zero. The forcing function f is a small scale random source with amplitude f_o which generates velocity fluctuations with a characteristic wavenumber k_f. The friction term ($\lambda\omega$) dissipates energy at some characteristic wavenumber k_d, which scales as $k_d \sim k_f f_o^{-3/4}(\lambda/k_f)^{3/2}$ in the passive scalar case. This friction term is equivalent to one proposed by Paret and Tabeling (1997) to parametrize the large-scale dissipation in their magnetically driven two-dimensional turbulence experiments. A linear friction term also arises in two-dimensional magnetohydrodynamic turbulence (Biskamp and Schwarz, 1997), in which case λ represents a particle collision frequency. As in Boffetta, *et. al.* (2000), the usual Laplacian term for the viscous dissipation is replaced by an eighth order ($n = 4$) hyperviscosity, adjusted so as to be negligible except on scales smaller than the forcing scale. The use of hyperviscosity is a standard numerical practice and should have no consequences in this work since our attention is focused on length scales larger than the energy injection scale.

The temperature field is advected according to the usual equation:

$$\partial_t T + \vec{u}\cdot\nabla T = \kappa\nabla^2 T. \quad (2)$$

An average temperature gradient g_o aligned with the gravitational acceleration is imposed by means of jump-periodic boundary conditions. The velocity boundary conditions are periodic.

The numerical resolution for all the results in this paper is 2048 by 2048 and the statistics are collected over at least 10 turnover times in each of our simulations. For the passive case the friction coefficient λ is chosen so that the dissipation scale Λ is approximately 1/10th of the box size. For non-zero $g\alpha$, the dissipation scale decreases from its passive scalar value. We completed two sets of simulations with different forcing scales; one with $k_f \sim 35$ and one with $k_f \sim 200$. Each set yielded similar conclusions; for brevity we only present graphical results from the $k_f \sim 200$ simulations.

Parameters

We now briefly discuss three parameters which characterize stably stratified flow in two and three dimensions: **(1)** Re: Three dimensional turbulence is characterized by its Reynolds number: $\mathrm{Re} \equiv U_\Lambda\Lambda/\nu$, where Λ is the length scale of the stirring, U_Λ is a typical velocity magnitude for motions of size Λ, and ν is the viscosity. Unlike three-dimensional turbulent flows, we stir the two-dimensional flow at large wavenumber (k_f) and the friction term dissipates energy at a smaller wavenumber k_d. The dissipation wavenumber k_d determines the 'integral scale' $\Lambda \equiv 2\pi/k_d$ in the two-dimensional inverse cascade spectrum. Thus in our flow the ratio of these scales $(k_f/k_d)^{4/3}$ determines the inertial range and characterizes the strength of the turbulence; it is therefore analogous to the three-dimensional Reynolds number. **(2)** Ri: In stably stratified turbulent shear flow, the influence of the stratification on the dynamics is characterized by the Richardson number, $\mathrm{Ri} \equiv g\alpha g_o\Lambda^2/U_\Lambda^2$, which gives the ratio of potential to kinetic energy on the lengthscale Λ. The same definition can be applied to our two-dimensional simulations, with Λ being the energy dissipation scale and the integral scale velocity being $U_\Lambda \equiv \sqrt{\langle u^2 + w^2\rangle}$. Because the energy input occurs at the small scales in our simulations, it is also useful to define a forcing-scale Richardson number, Ri_f, given by $\mathrm{Ri}_f \equiv g\alpha g_o/f_o k_f^2$. It indicates the ratio of potential to kinetic energy on the forcing scale k_f^{-1}. In our work, Ri_f is an input parameter defining the dimensionless stratification strength while Ri must be computed from the observed dissipation scale quantities Λ and U_Λ. **(3)** Pr: The ratio of the viscous dissipation scale to the thermal dissipation scale in three dimensions is determined by the Prandtl number, $\mathrm{Pr} \equiv \nu/\kappa$. In our simulations, k_f^{-1} plays the role of the viscous dissipation scale, and the molecular diffusivity κ is chosen so that temperature dissipation occurs at a larger length-

$\mathrm{Ri}_f \cdot 10^6$	Ri	$(k_f/k_d)^{4/3}$	$\sqrt{\langle\theta^2\rangle}$	U_Λ	$\Lambda(Ri_f)$
0	0	102.2	0.13	3.54	1.01
0.633	2.74	85.1	0.13	3.51	0.88
6.33	25.7	51.0	0.097	3.27	0.60
31.7	53.7	23.6	0.033	2.85	0.34
63.3	73.9	15.2	0.025	2.63	0.24
76.0	84.4	13.4	0.022	2.53	0.22

Table 1: Quantities for $k_f \sim 200$ simulations.

scale than k_f^{-1}. This corresponds to Pr < 1. The ratio of these length scales was not varied.

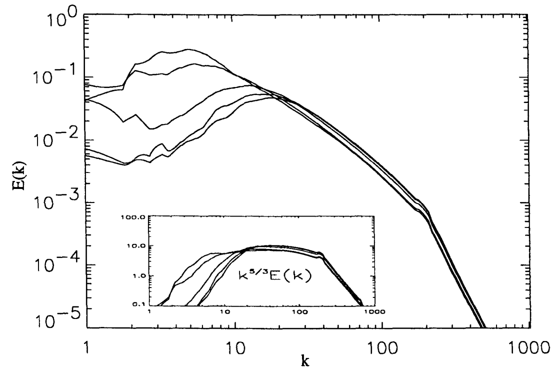

Figure 1: The kinetic energy spectra for the six stratification strengths listed in Table I. The peak value decreases as Ri_f increases. The inset shows the spectra compensated by $k^{5/3}$ to demonstrate the range of the inverse cascade, from $k_f = 200$ to $k_d(\mathrm{Ri}_f)$.

RESULTS

The dimensionless control parameter in our simulations is the forcing-scale Richardson number Ri_f. As Ri_f is increased, the integral-scale Richardson number Ri also increases while the integral scale Λ decreases. The effective Reynolds number $(k_f/k_d)^{4/3}$ also decreases, as does the integral scale velocity magnitude U_Λ. The precise values for the six simulations presented here (all with $k_f \sim 200$) are sumarized in Table I.

In the absence of gravity, the simulations produce two dimensional turbulence as investigated in Boffetta, *et. al.* (2000). The two-dimensional inverse kinetic energy cascade results in a $k^{-5/3}$ power spectrum within an 'inertial range' bounded by the large dissipation scale k_d^{-1} and the small forcing scale k_f^{-1}. Figure 1 shows kinetic energy spectra for the simulations listed in Table I. As the stratification strength Ri_f increases, the dissipation scale decreases and the inertial range shortens; the spectrum with the largest dissipation scale ($k_d \sim 10$) corresponds to $\mathrm{Ri}_f = 0$. One can see that the spectra are nearly identical at small scales and display the $k^{-5/3}$ power law in their respective inertial ranges. Scales much below the integral scale do not appear to be influenced by changes in the stratification strength. The principal dynamical effect of the stratification is therefore summarized by the decrease in the integral scale $\Lambda \equiv 2\pi/k_d$ as Ri_f increases. While a precise determination of k_d from the spectra is not possible, our estimates of the lower bound of the $k^{-5/3}$ scaling range in Figure 1 are consistent with

$$\Lambda(\mathrm{Ri}_f) = \frac{\Lambda(0)}{1 + 2000\mathrm{Ri}_f^{2/3}}. \quad (3)$$

The exponent 2/3 is a good approximation to the scaling of the integral scale, but is not a precise fit because the estimate of k_d is imprecise. However, with this integral scale it is possible to collapse the temperature statistics for various Ri_f to the same functional form for rare events. In our simulations $\Lambda(\mathrm{Ri}_f)$ varies by a factor of approximately four. The dynamical impact of the stratification in the largest Ri_f simulation is therefore quite significant compared to the passive scalar case.

The simplest statistical quantity examined in passive scalar flow is the fluctuation of the temperature from the mean gradient, defined as $\theta(\mathbf{x}) \equiv T(\mathbf{x}) - \mathbf{g_o} \cdot \mathbf{x}$. It has been observed that the probability distribution function $p(\theta)$ exhibits exponential tails, *i.e.*, $p(\theta) \sim \exp(-|\theta|/g_o\Lambda)$ for $|\theta| \gg g_o\Lambda$, in both grid turbulence (Gollub, *et. al.*, 1991, and Jayesh and Warhaft, 1991) and pipe flow (Guilkey, *et. al.*, 1997) experiments. A simple argument for exponential tails goes as follows: Since generation of fluctuations $|\theta| \gg g_o\Lambda$ requires the transport of fluid elements over many integral scales, and since fluid motions are incoherent on these length scales, fluid elements essentially undergo a random walk. The probability of taking a large number of sequential steps in the same direction is multiplicative, and this multiplication of independent probabilities yields the exponential PDF. Since the characteristic step size is the largest mixing length, one expects the PDFs to collapse when normalized by $g_o\Lambda(\mathrm{Ri}_f)$. Figure 2 demonstrates this collapse for several values of Ri_f, using equation 3 for Λ. The shape of the PDFs is rounded at the core and suggests exponential tails for $|\theta| > 2g_o\Lambda(\mathrm{Ri}_f)$. However, regardless of the precise form of the tails, the collapse when normalized by $g_o\Lambda(\mathrm{Ri}_f)$ is clear.

The statistical quantity most commonly studied in passive scalar flow is the two-point temperature difference, defined as $\delta\theta(\mathbf{r}) \equiv \theta(\mathbf{x}+\mathbf{r}) - \theta(\mathbf{x})$. The PDF of $\delta\theta$ quantifies the

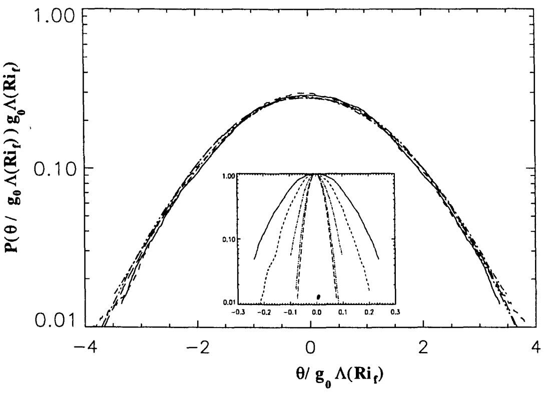

Figure 2: The PDFs of $\theta/g_o\Lambda(\mathrm{Ri}_f)$ for the six stratification strengths listed in Table I. The inset shows the uncollapsed PDFs of θ for the nonzero stratification strengths: the widest PDF (solid line) is for Ri = 2.74; the narrowest PDF (long-dashed line) is for Ri = 84.4.

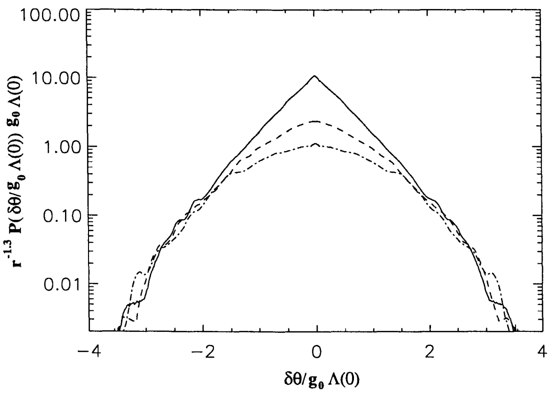

Figure 3: Collapse of the tails of the PDFs of temperature differences at various separations rk_f in the passive scalar case ($\mathrm{Ri}_f = 0$). The separations are (from top to bottom) rk_f = 30, 61, and 92. The rescaling exponent is $\zeta = 1.3$.

likelihood of observing a 'jump' in the temperature of $\delta\theta$ between two points separated by a distance $\mathbf{r}$. While less is known experimentally about the full PDF of $\delta\theta$, its moments (known as structure functions, $S_{2n}(r) \equiv \langle \delta\theta^{2n}(r) \rangle$ for moment $2n$) have been observed to scale with the separation r with exponents ζ_{2n} which are nonlinear functions of n (Antonia, *et. al.*, 1984, and Mydlarski and Warhaft, 1998). Based on two-dimensional simulations similar to ours, it was suggested by Celani, *et. al.*, (2000) that these exponents are due to an underlying PDF of the form

$$p(\delta\theta(r)) \sim r^{\zeta} Q(\delta\theta/g_o\Lambda) \qquad (4)$$

where $Q(\delta\theta/g_o\Lambda)$ is some universal function. This form implies the saturation of the exponents ζ_{2n} to a constant in the limit $n \to \infty$. Using a simple stochastic mixing model, it was suggested by Wunsch (1998) that $Q(x) \sim \exp(-|x|)$, and this model also led to the collapse of the exponents to a constant at large n.

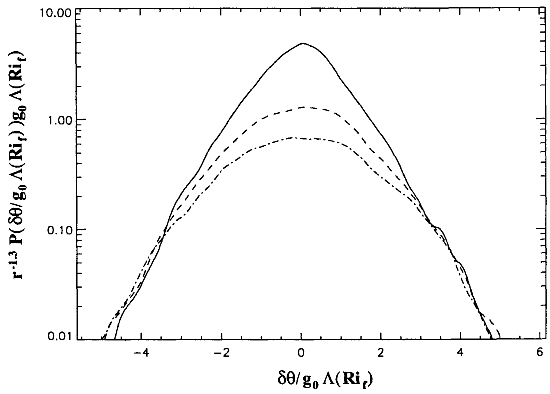

Figure 4: Collapse of the tails of the PDFs of temperature differences at various separations rk_f for the case Ri = 53.7. The separations are (from top to bottom) rk_f = 30, 61, and 92. The re-scaling exponent is $\zeta = 1.3$.

In this model, the largest mixing length determined the statistics of large $|\delta\theta|$ events according to the random walk argument given above, which applies to $\delta\theta$ as well as θ if the likelihood of a large difference is roughly equivalent to the likelihood of a large fluctuation at one of the two points. In this case, $Q(x)$ is independent of r. Consequently, if the large mixing length does control the statistics of large $|\delta\theta|$, one would expect the universal form of equation 4 to apply to the stably stratified case. Figure 3 shows this universal form for the passive scalar case ($\mathrm{Ri}_f = 0$) by rescaling the PDFs for different separations r, while Figure 4 demonstrates that it applies equally well for stable stratification (Ri = 54 in this case). Note that the cores do not overlap; the universal form applies only to the rare events. The shape suggests exponential tails for $Q(\delta\theta/g_o\Lambda)$. The exponent ζ is approximately 1.3 for all Ri_f. The saturation of the structure function scaling exponents inferred from this collapse in the passive scalar case also extends to our stably stratified turbulence simulations.

Figure 5 shows how the tails of the PDFs of $\delta\theta$ from simulations with different Ri_f collapse to the same form when normalized by the mixing length $\Lambda(\mathrm{Ri}_f)$. Again, the cores of the PDFs do not collapse but the tails reduce to a common, plausibly exponential, form. While the tails do not perfectly overlap, comparison with the inset reveals that most of the stratification dependence is accounted for by normalizing by the mixing length $\Lambda(\mathrm{Ri}_f)$. Taken together, Figures 3-5 demonstrate that the tails of the PDFs can be reduced to the same form for all inertial range separations r and all stratification strengths Ri_f. The dominance of the largest mixing length in determining the statistics of large temperature differences is the apparent reason for the universal form.

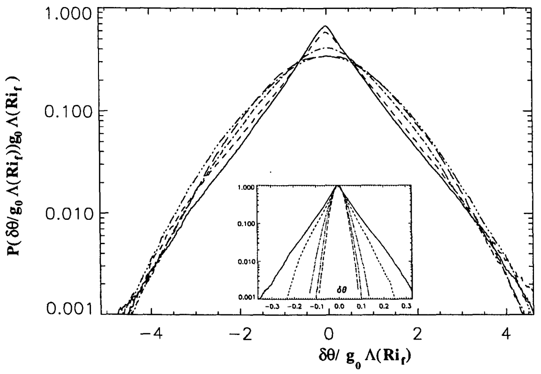

Figure 5: Collapse of the tails of the PDFs of $\delta\theta$ for the five nonzero stratification strengths Ri_f at the same separation $rk_f = 15$. The cores become more rounded as Ri_f increases. The inset shows the uncollapsed PDFs of $\delta\theta$, demonstrating the narrowing of the PDFs as Ri_f increases.

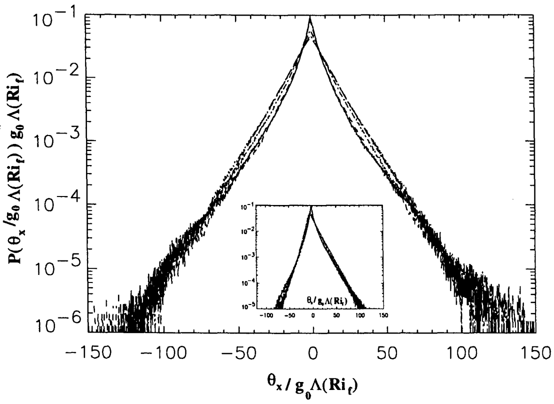

Figure 6: Collapse of the tails of the PDFs of the horizontal scalar gradient $\theta_x/g_o\Lambda$ for the five nonzero stratification strengths Ri_f. The inset shows the collapse the vertical scalar gradient, $\theta_z/g_o\Lambda$.

In the limit of zero separation ($r \to 0$), temperature differences reduce to temperature gradients. PDFs of temperature gradients, $\partial_x\theta$ and $\partial_z\theta$, have been observed to have exponential tails in stably stratified grid turbulence experiments (Thoroddsen and Atta, 1992). Again, we normalize our temperature gradient PDFs using the stratification-dependent lengthscale $\Lambda(\mathrm{Ri}_f)$ and observe that the tails collapse to the same exponential form. Figure 6 shows the collapse of the PDFs of $\partial_x\theta/g_o\Lambda$ for various stratification strengths.

CONCLUSIONS

The simulations presented here suggest that a variety of rare event statistics (temperature fluctuations, differences, and gradients) in stably stratified turbulence can be understood in terms of a single large mixing length in the flow. The stratification plays a direct role in determining this mixing length, which is inferred from the kinetic energy spectrum. Since the largest mixing length is the dominant factor in determining the likelihood of rare events, these statistics are identical to those of a passive scalar with the same mixing length. The universal form of the tails of the PDFs of temperature differences in the inertial range suggested by Celani, *et. al.*, (2000) therefore extends to stably stratified flows. As in the passive scalar case, the influence of the large scales is felt down to the dissipative range. Additionally, the rare event statistics of temperature fluctuations and gradients at various stratification strengths also exhibit a universal form when normalized by the same mixing length.

The precise dependence of the mixing length on the stratification strength in these two-dimensional simulations is not understood. Naively, one would expect the Ozmidov scale, $k_f\Lambda(\mathrm{Ri}_f) \sim \mathrm{Ri}_f^{-3/4}$, (the lengthscale at which the kinetic energy cascade balances the gravitational energy) to determine the mixing length. The approximate $\Lambda(\mathrm{Ri}_f) \sim \mathrm{Ri}_f^{-2/3}$ scaling observed here may be due to interactions with the large scale dissipation, which determines $\Lambda(0)$ and still plays a significant role in the energy dissipation even at the largest Ri_f we simulated. Consequently, this specific scaling of $\Lambda(\mathrm{Ri}_f)$ may not by asymptotic and may not apply to three dimensional turbulent flows. However, the argument that the stratification influences only the large scales, and that a single stratification-dependent mixing length determines the statistics of rare events at all scales, plausibly extends to three dimensions. Experiments should be capable of testing this hypothesis by comparing PDFs at various stratification strengths when normalized by the largest length scale in the kinetic energy spectrum.

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