NONUNIFORM STRETCHED WEIGHT IN LINE STATISTICS OF TURBULENCE

Susumu Goto
Theory and Computer Simulation Center,
National Institute for Fusion Science,
322-6 Oroshi-cho, Toki, 509-5292, Japan
goto@toki.theory.nifs.ac.jp

Shigeo Kida
Theory and Computer Simulation Center,
National Institute for Fusion Science,
322-6 Oroshi-cho, Toki, 509-5292, Japan
kida@toki.theory.nifs.ac.jp

ABSTRACT

Statistics of passive material lines in homogeneous isotropic turbulence are investigated by the use of direct numerical simulations. Passive material lines are stretched locally by the Kolmogorov-scale vortices, and their total length increases exponentially in time with the stretching rate of 0.17 (Kolmogorov time)$^{-1}$ at the Taylor-micro-scale Reynolds numbers of 56 and 84. It is shown theoretically and numerically that the commonly used simple arithmetic mean of stretching rates of many infinitesimal line elements underestimates the real stretching rate of passive material lines owning to nonuniform stretching along the lines.

INTRODUCTION

It is fundamental in turbulence phenomena how material objects in turbulence evolve temporally. The advection of material lines in turbulence has been extensively studied by many authors with relation to vortex line stretching. Dynamics of material surfaces are also important in turbulent mixing because they may be regarded as boundaries between two kinds of fluids. Furthermore, the advection and deformation of floating material objects in turbulence can be a powerful tool to visualize flow fields both in experiments and numerical simulations.

For the purpose of investigating statistical aspects of material objects, a set of infinitesimal material line (or surface) elements have been frequently used. For example, the stretching rate of material lines has been estimated by the simple arithmetic average of stretching rates of many material line elements (Girimaji and Pope 1990, Huang 1996). However, the equi-weight line-element statistics are different from the line statistics which have homogeneous weight along material lines. This is because one-dimensional objects are nonuniformly stretched even in a three-dimensionally incompressible flow. The aim of this paper is to emphasize important roles of the nonuniform statistical weight in the line statistics.

PASSIVE LINE IN TURBULENCE

Governing Equations

We consider a material line advected by a turbulent flow according as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{x}_l(t) = \boldsymbol{u}(\boldsymbol{x}_l(t), t)\,. \qquad (1)$$

Here, $\boldsymbol{x}_l$ denotes a position of a point labeled with l on the line, and $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity of an incompressible fluid at position $\boldsymbol{x}$ at time t. Equation (1) implies that the line is advected together with fluid elements. On the other hand, dynamics of the velocity field are not affected by the material line. In this sense, this line is called the passive material line. The velocity field is governed by the Navier-Stokes equation,

$$\frac{\mathrm{D}}{\mathrm{D}t}\,\boldsymbol{u}(\boldsymbol{x}, t) = -\,\frac{1}{\rho}\,\nabla p(\boldsymbol{x}, t) + \nu\,\nabla^2\,\boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{f}(\boldsymbol{x}, t) \qquad (2)$$

and the equation of continuity,

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0 \,, \tag{3}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla \tag{4}$$

stands for the Lagrangian derivative. Here, ρ is the constant density of the fluid, ν is the kinematic viscosity, $p(\boldsymbol{x}, t)$ is the pressure and $\boldsymbol{f}(\boldsymbol{x}, t)$ is an external force, which is introduced to make turbulence statistically stationary.

Stretching Rate

A typical temporal evolution of a passive material line in statistically homogeneous isotropic turbulence in a periodic box is shown in Fig.1, where the Taylor-micro-scale Reynolds number R_λ is 84. This is a numerical result described later. The passive material line is strongly stretched by turbulence and spreads very rapidly in the whole box. The total length $L(t)$ of the line increases exponentially as

$$L(t) = L(0) \exp\Big[\gamma t\Big] \,, \tag{5}$$

where γ is the stretching rate. Since Batchelor (1952), many efforts have been made to estimate γ theoretically or numerically. We are also aiming at a correct and precise estimation of the stretching rate. The time evolution of the total length of passive material lines is plotted in Fig.2 in a semi-logarithmic scale for two different Reynolds numbers $R_\lambda = 56$ and 84. Here, the time is normalized by the temporal average of the Kolmogorov time,

$$\tau_\eta = \epsilon^{-\frac{1}{2}} \nu^{\frac{1}{2}} \,, \tag{6}$$

with ϵ being the the energy dissipation per unit mass. The coincidence of the two lines implies that the Kolmogorov similarity should be well satisfied at these Reynolds numbers. The stretching rate estimated from the slopes of the lines is around

$$\gamma = 0.17\, (\tau_\eta)^{-1} \,. \tag{7}$$

This value is larger, by about 30%, than $0.13 \sim 0.14(\tau_\eta)^{-1}$ estimated from the arithmetic mean of stretching rates of many infinitesimal passive material line elements in isotropic turbulence at comparable Reynolds numbers (Girimaji and Pope 1990, Huang 1996). In the followings, we shall describe the reason of their underestimation.

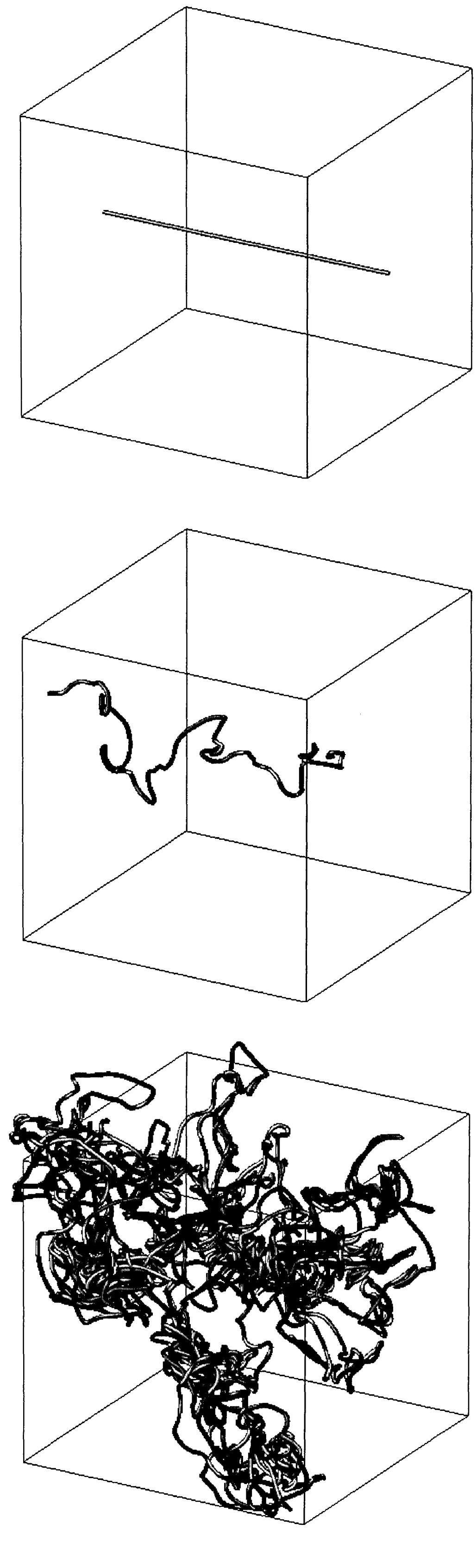

Figure 1: Time evolution of a passive material line in homogeneous isotropic turbulence. $t = 0$, 1 ($7.1\tau_\eta$), and 4 ($28\tau_\eta$). The Taylor-micro-scale Reynolds number R_λ is 84.

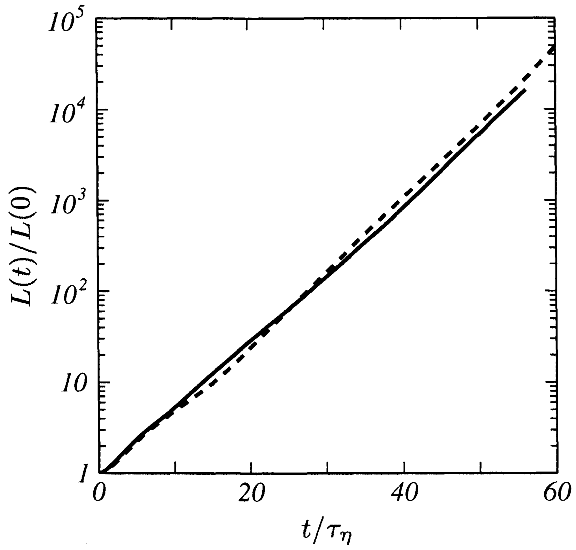

Figure 2: Exponential increase of total length of passive lines. Solid line: $R_\lambda = 56$; dashed line: 84.

NONUNIFORM STRETCHED WEIGHT

Line Average and Line-Element Average

Suppose that a passive material line consists of I line segments, each of which has sufficiently short length $\Delta l^{(i)}(t)$ $(i = 1, 2, \cdots, I)$ so that the total length of the line may be expressed by the sum of $\Delta l^{(i)}(t)$ as

$$L = \sum_{i=1}^{I} \Delta l^{(i)} . \qquad (8)$$

Then, the stretching rate defined by (5) is expressed as

$$\gamma = \frac{1}{L} \sum_{i=1}^{I} \gamma_e^{(i)} \Delta l^{(i)} \left(= \frac{\int \gamma_e \, \mathrm{d}l}{\int \mathrm{d}l} \right) \qquad (9)$$

in terms of the stretching rate of each line segment,

$$\gamma_e^{(i)} = \frac{1}{\Delta l^{(i)}} \frac{\mathrm{d}\Delta l^{(i)}}{\mathrm{d}t} . \qquad (10)$$

The right-hand side of (9) implies that the stretching rate γ of the passive material line is nothing but the average of line-element stretching rates γ_e along the line, that is,

$$\gamma = \langle \gamma_e \rangle_{\text{line}} . \qquad (11)$$

However, the true value γ of the stretching rate is generally different from the simple arithmetic average,

$$\langle \gamma_e \rangle_{\text{line-element}} = \frac{1}{I} \sum_{i=1}^{I} \gamma^{(i)} \qquad (12)$$

of γ_e over a set of many line elements (the line-element average), that is,

$$\langle \gamma_e \rangle_{\text{line-element}} \neq \langle \gamma_e \rangle_{\text{line}} . \qquad (13)$$

Note that these two averages are identical, if all the line segments have equal length.

Stretched Weight Factor

Let us assume, without loss of generality, that all the line segments have a same length at the initial time, i.e.,

$$\Delta l^{(i)}(0) = \Delta l_0 . \qquad (14)$$

Then, each segment can be treated equivalently in the line statistics at least initially. However, at a later time, they will generally get nonequivalent because of nonuniform stretching, and the stretched weight factor,

$$\sigma^{(i)}(t) = \frac{\Delta l^{(i)}(t)}{\Delta l_0} = \exp\left[\int_0^t \gamma_e^{(i)}(t') \, \mathrm{d}t' \right] \qquad (15)$$

must be taken into account. Rewriting (9) as

$$\gamma = \frac{\sum_{i=1}^{I} \gamma_e^{(i)} \sigma^{(i)}}{\sum_{i=1}^{I} \sigma^{(i)}} = \frac{\langle \gamma_e \sigma \rangle_{\text{line-element}}}{\langle \sigma \rangle_{\text{line-element}}} , \qquad (16)$$

we can see that the line average and the line-element average are related with each other through the stretched weight factor σ.

For a later discussion, let us recall that the arithmetic mean of stretching rates of many infinitesimal line elements has been often employed to estimate the stretching rate of passive material lines numerically (Girimaji and Pope 1990, Huang 1996). The above discussion, however, warns that all the line elements cannot be treated with equi-statistical weight, even though they are always infinitesimal, and the stretched weight factor is necessary to be introduced.

In general, the line average of a quantity g accompanied with passive material lines is expressed in terms of averages over the constituent line elements as

$$\langle g \rangle_{\text{line}} = \frac{\langle g\sigma \rangle_{\text{line-element}}}{\langle \sigma \rangle_{\text{line-element}}} . \qquad (17)$$

Hence, if g and σ are statistically independent of each other, then the line average and the line-element average of g are identical. If, on the other hand, they are correlated (as in the case of γ_e and σ), these two averages give different answers. Since the stretching rate and the stretched weight factor are expected to be positively correlated, the line average of γ_e should be larger than the line-element average.

DIRECT NUMERICAL SIMULATION

In this section, we give a numerical evidence to confirm the theoretical prediction, described in the preceding section, that the line average and the line-element average are essentially different from each other and they are connected through the stretched weight factor σ.

Numerical Method

The advection equation (1) and the Navier-Stokes equation (2) are solved simultaneously by numerical integration. The Fourier spectral method is employed for spatial derivatives and the 4-th order Runge-Kutta-Gill method for time derivatives. The aliasing interactions are removed by the phase shift method. The amplitudes of the Fourier components of velocity of wavenumbers smaller than $k_f(= 2.5)$ are kept constant, while their phases evolve temporally according to the governing equation. This replaces an effective forcing $\boldsymbol{f}(\boldsymbol{x}, t)$ in large scales. The right-hand side of (1) is estimated by the 4^3-point Lagrangian interpolation of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ at the grid points.

We carried out two kinds of numerical simulations: The line simulation and the line-element simulation. In the former, the passive material line consists of I segments, i.e., $I + 1$ nodes, and the position $\boldsymbol{x}^{(i)}(t)$ $(i = 0, 1, \cdots, I)$ of each node is governed by (1). To keep the numerical accuracy against the stretching of the line, the distance between a pair of successive nodes are always kept much smaller than the Kolmogorov length by adding a new node at the center of a segment as soon as it becomes longer than a threshold (1.5 times numerical grid, say). This line simulation gives the line average of any quantity accompanied with lines. In the latter line-element simulation, we track many passive material line segments, each of which consists of pairs of two passively advected points with sufficiently close distance. In the same way as the line simulation, the position of a passive point is advected according to (1). At every numerical time step, the length of each line segment is renormalized to be kept as short as the numerical grids. This line-element simulation, which is essentially equivalent to those simulations of many infinitesimal passive material line elements, leads to the line-element average.

We present here numerical results of two different sets of parameters: The kinematic viscosities ν are 5×10^{-3} and 2.5×10^{-3}, the resolutions N^3 are 128^3 and 256^3, the Taylor-micro-scale Reynolds numbers R_λ are 56 and 84 on the temporal average, the Kolmogorov time τ_η are 2.0×10^{-1} and 1.4×10^{-1} on the average, respectively. These Reynolds numbers are small enough that the smallest-scale turbulent motions, which play crucial roles in the stretching of passive material lines, are well resolved in the both cases.

Numerical Results

In each of the line simulation and the line-element simulation, we keep the number of line segments equal to N^3 in order to fix the sample number in averaging. In the line simulation, an end-point of a passive material line is discarded every time a new node is added to the line according to the algorithm described in the preceding section. The line average and the line-element average of the stretching rate of passive lines are then estimated by the line simulation and the line-element simulation, respectively. We repeat each of these simulations for ten different turbulent flows, and the mean stretching rates obtained by the line average and the line-element average are plotted in Fig.3(a) with thick and thin curves, respectively. The gray zones denote standard deviations for the ten realizations. The mean stretching rates start at the origin, increase in time, and settle down to constant values in a later stage. The initial vanishment of the stretching rate may be understood as follows. The line-element stretching rate γ_e may be expressed in terms of the eigenvalues s_1, s_2, s_3 of the rate-of-strain tensor $\mathbf{S}$ and the angles α_1, α_2, α_3 between the line segment vector $\Delta \boldsymbol{l}$ and each of the three principal axes of $\mathbf{S}$ as

$$\gamma_e = \frac{\Delta \boldsymbol{l} \cdot \mathbf{S} \cdot \Delta \boldsymbol{l}}{\Delta l^2} = s_1 \cos\alpha_1 + s_2 \cos\alpha_2 + s_3 \cos\alpha_3 \,. \quad (18)$$

If the line segments distribute isotropically independent of the rate-of-strain tensor, then $\langle\cos\alpha_1\rangle = \langle\cos\alpha_2\rangle = \langle\cos\alpha_3\rangle$ and the averaged stretching rate vanishes because the fluid is incompressible $(s_1 + s_2 + s_3 = 0)$. This is

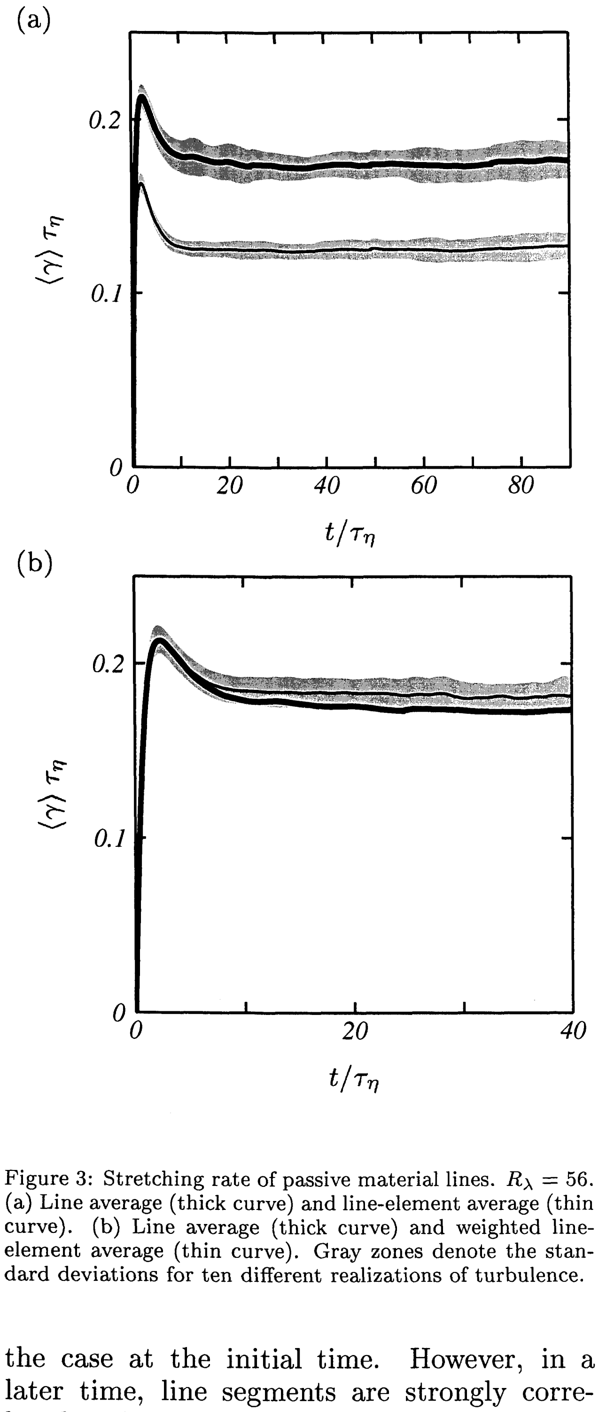

Figure 3: Stretching rate of passive material lines. $R_\lambda = 56$. (a) Line average (thick curve) and line-element average (thin curve). (b) Line average (thick curve) and weighted line-element average (thin curve). Gray zones denote the standard deviations for ten different realizations of turbulence.

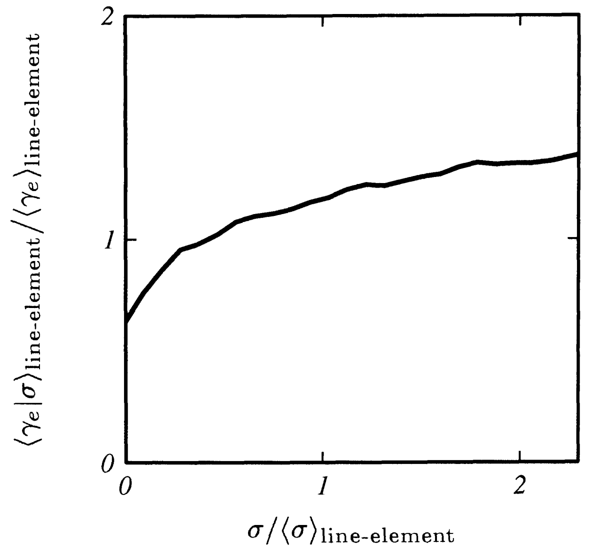

Figure 4: Conditional line-element average of the stretching rate for a given stretched weight factor. $R_\lambda = 56$, $t = 20\tau_\eta$.

the case at the initial time. However, in a later time, line segments are strongly correlated with the rate-of-strain tensor $\mathbf{S}$, and the averaged stretching rate does not vanish.

The mean values of the stretching rate averaged over a period ($20\tau_\eta \le t \le 90\tau_\eta$) in the statistically stationary state are

$$\langle\gamma\rangle_{\text{line}} = (0.17 \pm 0.01)\ {\tau_\eta}^{-1}, \qquad (19)$$

and

$$\langle\gamma\rangle_{\text{line-element}} = (0.13 \pm 0.01)\ {\tau_\eta}^{-1} \qquad (20)$$

in the line average and the line-element averages, respectively. Here, $\pm$ denotes the temporal average of the standard deviation for ten different turbulent flows. As expected, the line average (19) coincides with (7), whereas the line-element average (20) with the value $0.13 \sim 0.14{\tau_\eta}^{-1}$ estimated before by the simple arithmetic average of stretching rates of many infinitesimal line elements. The fact that the line average is larger than the line-element average implies the positive correlation between the stretching rate γ_e of the line element and the stretched factor σ defined by (15) (see the relation (16)). This positive correlation between γ_e and σ might be intuitively obvious because σ is an integrated stretching rate. In Fig.4, we plot the conditional line-element average $\langle\gamma_e|\sigma\rangle_{\text{line-element}}$ of the stretching rate for a given value of σ at $t = 20\tau_\eta$. It is seen that γ_e and σ are indeed positively correlated.

Finally, we calculate numerically the line-element average (16) taking the nonuniform stretched weight into account. This weighted line-element average should retrieve the line average in principle. For this purpose, we calculate the stretched weight factor σ of each line segment in addition to the stretching rate in the line-element simulations. The weighted line-element average (16) thus obtained is plotted in Fig.3(b) with a thin curve and compared with the line average (thick curve). The weighted line-element average indeed coincides with the line average in the beginning ($t \lesssim 5\tau_\eta$), but two curves deviate from each other in later times. The temporal mean of the weighted line-element average in the statistically stationary state ($t \geq 20\tau_\eta$) is

$$\langle\gamma\rangle_{\text{weighted-line-element}} = 0.18\ {\tau_\eta}^{-1}\ , \qquad (21)$$

which is larger than the true value by about 6%. This overestimation of the weighted line-element average originates from a drawback inherent in the line-element simulation. That is, adjacent line elements separate from each other quite rapidly. Because of this rapid separation, a continuous passive line cannot be tracked by line-element simulations after several Kolmogorov times when the mean stretching rate enters the statistically stationary state. Recall that the formula (16) is based upon the assumption that many line segments constitute a complete passive line. Thus, the line statistics can be retrieved only in the early stage of evolution by the line-element simulation. More precisely, the line-element simulations overestimate the stretched weight factor σ and therefore γ_e in later times (see Kida and Goto 2001 for more detailed discussion).

CONCLUDING REMARKS

An appropriate numerical method has been proposed to analyse the statistics of passive material lines. In order to estimate the average of any physical quantity along passive material lines correctly, the lines must be tracked faithfully. A good answer cannot be obtained by those numerical simulations of many infinitesimal line elements employed by previous authors. As a typical example, we examined the stretching rate of passive material lines. The simple arithmetic average over many line elements underestimates the true value by about 23% (Fig.3(a)) because of neglection of the nonuniform stretching along passive material lines. On the other hand, the weighted average by the stretched factor overestimates it by about 6% (Fig.3(b)) because of the inaccurate representation of passive material lines in the line-element simulations.

It should be stressed again that the difference between the line average and the line-element average of the stretching rate is brought about by the correlation between the stretching rate γ_e and the stretched weight factor σ (Fig.4). A passive material line is trapped by vortices of the Kolmogorov length scale, and is stretched by the strain around them. The effect of this trapping may bring the positive correlation between σ and γ. From a number of direct numerical simulations, we have learned that there exist coherent structures in turbulence, e.g., the "elementally vortex" (Kida 2001). However, the computer power is not strong enough to reveal the Reynolds number dependence of such micro-scale structures. Although we have seen that the Kolmogorov similarity of the stretching of passive material lines does approximately hold at $R_\lambda = 56 \sim 84$ (Fig.2), there is a possibility that it would break down due to a kind of mixing transition of the micro-scale structures around $R_\lambda \approx 100 \sim 140$ (Dimotakis 2000). These Reynolds number dependence of turbulence structures and the line stretching is one of the quite interesting near-future problems.

This work has been partially supported by Grant-in-Aid for Scientific Research on Priority Areas (B) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

Batchelor, G. K., 1952, "The effect of homogeneous turbulence on material lines and surfaces," Proc. Roy. Soc. London A, 213, pp.349–366.

Dimotakis, P., 2000, "The mixing transition in turbulent flows," J. Fluid Mech., 409, pp.69–98.

Girimaji, S. S., and Pope, S. B., 1990, "Material-element deformation in isotropic turbulence," J. Fluid Mech., 220, pp.427–458.

Huang, M-J., 1996, "Correlations of vorticity and material line elements with strain in decaying turbulence," Phys. Fluids, 8, pp.2203–2214.

Kida, S., 2001, "Vortical Structure of Turbulence," *Proceedings, XX International Congress of Theoretical and Applied Mechanics*, Aref, H., ed., Kluwer Acad. Publ. (in press).

Kida, S., and Goto, S., 2001, "Line statistics: Stretching rate of passive lines in turbulence," (submitted; preprint, NIFS report 684).