USING STRUCTURE INFORMATION IN MODELING OF MAGNETOHYDRODYNAMIC TURBULENCE

Ola Widlund*
EPM–MADYLAM
ENSHMG, B.P. 95
F-38402 St Martin d'Hères, France
widlund@polycnrs-gre.fr

ABSTRACT

The properties of the magnetohydrodynamic turbulence closure proposed by Widlund *et al.* (1998) are discussed for the case of homogeneous turbulence decaying in a static magnetic field (at low magnetic Reynolds numbers). The model is a Reynolds stress closure, extended with a transport equation for a dimensionality anisotropy variable α, which carries information about length scale anisotropy. The model is shown to be consistent with theory and experiments for both weak and strong magnetic fields. For the initial linear decay in strong magnetic fields, it produces the $K \sim t^{-1/2}$ energy decay predicted by linear theory. When nonlinear effects are important, the model predicts an energy decay $K \sim t^{-1.7}$ and length scale evolution in agreement with the experiments of Alemany *et al.* (1979).

INTRODUCTION

Magnetic fields have found widespread use in many materials processing applications. In continuous casting of steel, for example, electrostatic magnetic fields are used to brake and control the mean flow of liquid metal in the mold. The magnetic field also causes magnetic Joule dissipation of turbulence, thus affecting turbulent transport of heat and mass. At the same time, turbulent structures tend to be elongated in the direction of the magnetic field. Numerical simulations of this and other turbulent magnetohydrodynamic (MHD) flows generally suffer from the inability of conventional turbulence models (like the K-ε model, or a full Reynolds stress model) to deal with the large anisotropies of length scales encountered in MHD turbulence. To improve the situation, it has been suggested to include information about length-scale anisotropy in an extended Reynolds stress model for MHD applications (Widlund *et al.*, 1998 and 2000).

The so-called dimensionality tensor, Y_{ij}, was first introduced by Reynolds (1989) and coworkers to help describe the effect of rapid rotation on turbulence. The dimensionality tensor is defined in physical and spectral space as

$$Y_{ij} \equiv -\frac{1}{4\pi}\int \frac{\partial^2 \overline{u_n u_n'}}{\partial r_i \partial r_j}\frac{dV'}{r} = \int \frac{k_i k_j}{k^2}\hat{E}_{nn} d^3\mathbf{k},$$

where $\hat{E}_{ij} \equiv \overline{\hat{u}_i^* \hat{u}_j}$ is the spectral energy tensor, $\overline{u_i u_j'}$ the corresponding two-point velocity correlation, and $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. While the Reynolds stress tensor R_{ij} accounts for the kinetic energy of fluctuations in different directions, the dimensionality tensor carries information about the length-scales in different directions. The latter information seems vital for a correct description of magnetic Joule dissipation of turbulence. For homogeneous turbulence, the Joule dissipation of turbulent kinetic energy can, for example, be exactly expressed in terms of the component of the dimensionality tensor which is parallel with the magnetic field,

$$\mu = \frac{1}{2}\mu_{nn} = \frac{\sigma B^2}{\rho} n_i n_j Y_{ji}, \qquad (1)$$

where n_i is a unit direction vector of the magnetic field.

Inspired by (1), Widlund *et al.* (1998) proposed to extend a conventional Reynolds stress closure with a model transport equation for a dimensionality anisotropy variable, α, defined as

$$\alpha \equiv \frac{n_i n_j Y_{ji}}{2K}, \quad 0 \le \alpha \le 1. \qquad (2)$$

$\alpha = 1/3$ for isotropic turbulence, and $\alpha = 0$ in the limit of 2D turbulence, when turbulent structures have grown very long in the direction of the magnetic field. The new variable

*Most of the work described here was performed by the author while at the Faxén Laboratory, KTH, Stockholm.

allowed the scalar Joule dissipation to be expressed exactly as

$$\mu = \frac{\sigma B^2}{\rho} 2K\alpha, \tag{3}$$

while the anisotropic Joule dissipation tensor μ_{ij} could be modeled with an invariant tensor function in R_{ij} and α.

An exact transport equation for α can be derived from the Navier–Stokes equations, including magnetic and inertial effects, as well as effects of mean shear and strain (Widlund *et al.*, 2000). All terms in the equation require modeling, however.

MODEL EQUATIONS

Let us consider here the case of homogeneous shear-free turbulence, initially axisymmetric about the magnetic field vector. We assume Cartesian coordinates, such that the magnetic field vector is in the x_3-direction, i.e. $\mathbf{B} = B\mathbf{e}_3$. Due to axisymmetry, there is only two independent Reynolds stress components, R_{11} and R_{33}, and the turbulent kinetic energy is given by $K = (2R_{11} + R_{33})/2$. If we introduce a magnetic time scale $\tau = \rho/(2\sigma B^2)$, the Reynolds stress closure proposed by Widlund *et al.* (1998) reduces to

$$\begin{aligned}
\frac{dR_{33}}{dt} &= \pi_{33} - \frac{2}{3}\varepsilon - \mu_{33}, && (4)\\
\frac{dK}{dt} &= -\varepsilon - \frac{1}{\tau}\alpha K, && (5)\\
\frac{d\varepsilon}{dt} &= -C_{\varepsilon 2}\frac{\varepsilon^2}{K} - \mu_\varepsilon, && (6)\\
\frac{d\alpha}{dt} &= \pi_\alpha - \mu_\alpha. && (7)
\end{aligned}$$

An equation for R_{11} is redundant, as it can be computed from K and R_{33}.

In (4), π_{33} is a component of a return-to-isotropy model for the slow pressure–strain rate term, while

$$\mu_{33} = \frac{R_{33}}{\tau}\left(\alpha + \frac{9}{10}\alpha^2\left[\frac{R_{33}}{K} - 2\right]\right) \tag{8}$$

is due to an invariant tensor model for the anisotropy of the Joule dissipation tensor (see Widlund *et al.* [1998] for details). Note that the K equation (5) is exact, as the Joule dissipation term follows from the definition of α. The first term in (6) is the standard model of viscous destruction of ε. The second term, modeled in the same spirit, represents magnetic destruction,

$$\mu_\varepsilon = \frac{C_{\varepsilon\alpha}}{\tau}\alpha\varepsilon. \tag{9}$$

In (7), μ_α is a magnetic (Joule) destruction term, driving the turbulence towards two-dimensionality ($\alpha \to 0$). π_α is a "return-to-isotropy" term due to nonlinear effects, which tends to restore turbulence to an isotropic state. We will see that the dynamical properties of the closure is largely determined by the properties of μ_α and π_α. The simplest models of these terms are

$$\begin{aligned}
\pi_\alpha &= C_{\alpha 2}\frac{\varepsilon}{K}\left(\frac{1}{3} - \alpha\right), && (10)\\
\mu_\alpha &= \frac{C_{\alpha 1}}{\tau}\alpha^2. && (11)
\end{aligned}$$

This assumes that the nonlinear effects are governed by a turbulent time scale K/ε, and that the magnitude of the return-to-isotropy term is proportional to the level of anisotropy. The magnetic term μ_α is modeled in analogy with the destruction terms in the K and ε equations.

With one-point closures for ordinary hydrodynamic turbulence, an integral length scale for the turbulence is usually estimated as

$$L \sim \frac{K^{3/2}}{\varepsilon}. \tag{12}$$

In the MHD case, length scales are anisotropic, as magnetic effects will make turbulent structures grow in the direction of the magnetic field. In this case we let L be a measure of the length scale *perpendicular* to the magnetic field. L further retains its role as a length scale related to eddies with turn-over time K/ε (this dual interpretation can be argued convincingly from a spectral analysis of the magnetic effects in Fourier space). Comparison between (3) and scaling laws put forward by other authors, e.g. Davidson (1997), suggests that

$$\alpha \sim \left(\frac{L}{L_\parallel}\right)^2, \tag{13}$$

where $L_\parallel$ is a characteristic length scale in the direction *parallel* with the magnetic field. Since $\alpha = 1/3$ for isotropic turbulence (for which we assume $L = L_\parallel$), we can thus estimate $L_\parallel$ in terms of L (for $L_\parallel \geq L$),

$$L_\parallel \approx \frac{L}{\sqrt{3\alpha}}. \tag{14}$$

MODEL PROPERTIES

The relative importance of magnetic effects can be characterized in terms of the magnetic interaction parameter N, which can be defined as the ratio of the turbulent and the magnetic time scales,

$$N \equiv \frac{\sigma B^2 K}{\rho \varepsilon}. \tag{15}$$

In the limit of large interaction parameters ($N \gg 1$), magnetic dissipation dominates over viscous dissipation and inertial effects (at least initially).

For $N \ll 1$, magnetic effects are negligible, and for $N = 0$ the model equations of the closure coincide with conventional K-ε or Reynolds stress closures. For homogeneous decaying turbulence, these closures predict an asymptotic energy decay $K \sim t^{-n}$, where $n \equiv 1/(C_{\varepsilon 2} - 1)$. We recall that the standard value $C_{\varepsilon 2} = 1.92$ gives $n \approx 1.09$, for agreement with experiments on grid generated turbulence.

Large N, and linear decay

For sufficiently large interaction numbers ($N \gg 1$) Joule dissipation dominates, so that viscous dissipation and nonlinear effects can be neglected. In a spectral analysis, the evolution of the spectral energy tensor is then described by Lehnert's (1955) linearized equations, and Moffatt (1967) predicted the asymptotic energy decay rate $K \sim t^{-1/2}$ for this regime of *linear decay*.

The MHD model equations have analytic solutions for this case, with an asymptotic energy decay rate given by $K \sim t^{-1/C_{\alpha 1}}$. The decay rate is thus controlled by the model coefficient $C_{\alpha 1}$ in the magnetic destruction term (11) of the α equation. By choosing $C_{\alpha 1} = 2$, the model is made consistent with the theoretical predictions. Widlund *et al.* (2000) used the linearized equations and the concept of rapid distortion theory (RDT) to compute the exact destruction term, for comparison with the model term (11). Figure 1 shows the ratio $\mu_\alpha \tau / \alpha$, as predicted by RDT and the simple model (11). A coefficient $C_{\alpha 1} = 0.8$ gives the correct value for isotropic turbulence ($\alpha = 1/3$), while $C_{\alpha 1} = 2$ gives a good description of the approach to the 2D limit. The energy decay and evolution of α predicted by RDT and the model closure are plotted in Fig. 2, for both values of $C_{\alpha 1}$. $C_{\alpha 1} = 0.8$ gives better agreement for small times, but a correct asymptotic energy decay requires $C_{\alpha 1} = 2$. The latter value is used in the following. Based on the RDT data in Fig. 1, it should be possible to device a more accurate higher-order model for μ_α, but this is beyond the scope of the present paper.

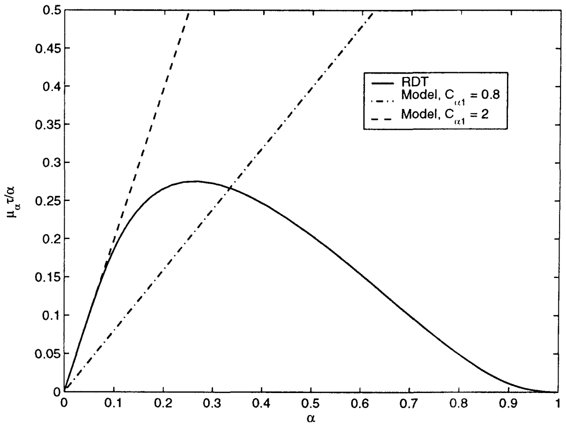

Figure 1: The non-dimensional ratio $\mu_\alpha \tau / \alpha$ as a function of α, as predicted by RDT and the model term (11) for two values of the coefficient $C_{\alpha 1}$.

The predicted integral length scale L can be shown to evolve as $L \sim t^{(C_{\varepsilon\alpha} - 3/2)/C_{\alpha 1}}$ for large times. It is generally held that a length scale perpendicular to the magnetic field should not be affected by the magnetic field. This is accomplished by choosing $C_{\varepsilon\alpha} = 3/2$. Furthermore, $\alpha \sim t^{-1}$ for large t, which through (14) suggests that the parallel length scale evolves as $L_\| \sim t^{1/2}$, as proposed earlier by Davidson (1997).

Nonlinear decay

For interaction parameters of order unity, the situation is complicated by the presence of viscous and nonlinear effects. Even for initially large interaction parameters, the effective Joule dissipation rate decreases with time, as α becomes smaller. We will therefor eventually reach a point where nonlinear effects can no longer be neglected (provided there is sufficient energy left). As explained in spectral terms by Alemany *et al.* (1979), the *nonlinear decay* that follows is due to a balance between Joule dissipation and nonlinear angular energy transfer. The situation is illustrated in Fig. 3. The Joule dissipation term (3) suggests that an effective time scale of the Joule dissipation is τ/α, rather than τ. As pointed out by Sreenivasan and Alboussière (2000), the effective Joule time scale and the turbulent time scale together form a "true" interaction parameter. We here define it using variables of the closure,

$$N^* \equiv \frac{\alpha K}{2\tau\varepsilon} = \frac{\sigma B^2 \alpha K}{\rho \varepsilon} = \alpha N. \tag{16}$$

Sreenivasan and Alboussière proposed that the nonlinear decay is characterized by a constant N^* of order unity.

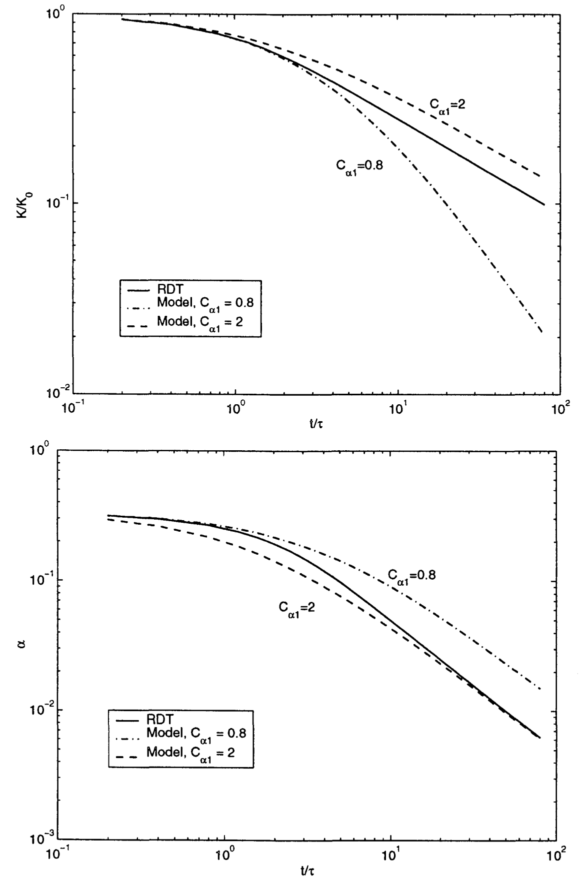

Figure 2: Decaying turbulence in the limit of large N, where Joule dissipation dominates. The graphs show time development ($\tau = \rho/(2\sigma B^2)$) of turbulent kinetic energy (top) and anisotropy α (bottom). Solid lines represent the RDT solution, and dashed lines the analytic solution of the model equations (5) and (7), for two values of $C_{\alpha 1}$. (Initially isotropic turbulence, $\alpha_0 = 1/3$.)

In the MHD turbulence closure, angular energy transfer is represented by the return-to-isotropy term π_α in the α equation. The behavior of π_α as we approach $\alpha = 0$ has been found to have a large impact on the model properties. In particular, the simple model term (10) is not sufficient. Its magnitude increases as α decreases, and attains its maximum value for $\alpha = 0$. In contrast, the true return term can be expected to first increase with increasing anisotropy, but then *decrease* towards zero as we approach the 2D limit ($\alpha = 0$). There are two important reasons to expect this behavior:

1. **2D dynamics:** The dynamics of 2D turbulence is decidedly different from that of 3D turbulence. One can argue that the spectral triad interactions responsible for the angular energy transfer should vanish in the limit of 2D turbulence, where all the energy is concentrated in the wave number plane $k_\parallel = 0$.

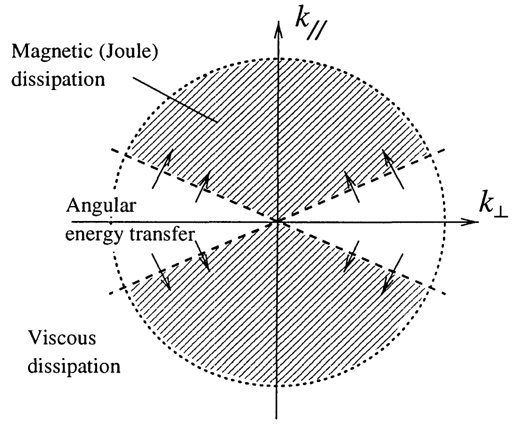

Figure 3: Illustration of the spectral energy distribution for the general nonlinear case. Joule dissipation is largest in a cone about an axis parallel with the magnetic field (shaded region). Viscous dissipation is significant outside the dotted circle, where wavenumbers are large (smallest scales). Nonlinear inertial mechanisms cause angular energy transfer from the energy-containing region near the $k_\perp$-plane, to the region where Joule dissipation is strong.

2. **Magnetic surpression of triple correlations:** Alemany *et al.* (1979) have suggested that the triple correlations responsible for nonlinear energy transfer are surpressed by a magnetic field. There are some evidence to this effect, both from DNS and experiments.

A rather crude attempt to address the first item above is to substitute the model term (10) with a modified version, which is piecewise linear in α,

$$\pi_\alpha = \frac{\varepsilon}{K} \min\left(C'_{\alpha 2}\alpha \; ; \; C_{\alpha 2}\left[\frac{1}{3} - \alpha\right]\right). \quad (17)$$

For small α, this model term behaves as $\pi_\alpha = C'_{\alpha 2}\alpha\varepsilon/K$. It can be shown that this property reflects the expected balance between angular energy transfer and Joule dissipation, and reproduces a constant N^* of order unity for the nonlinear phase of decay. One can argue that also the viscous terms in the K and ε equations should be modified in a similar way as we approach the 2D limit. If we restrict ourselves to a modification of π_α, however, the asymptotic energy decay rate will depend on $C'_{\alpha 2}$ in (17), as well as $C_{\alpha 1}$, $C_{\varepsilon\alpha}$ and $C_{\varepsilon 2}$. The latter three have already been given values to correctly predict the linear decay, and the conventional non-magnetic case. It is then found that a value of $C'_{\alpha 2} = 4.72$ reproduces the asymptotic energy decay rate $K \sim t^{-1.7}$ found experimentally by Alemany *et al.* (1979) for the nonlinear decay, while the length scales develop as $L \sim t^{0.15}$ and $L_\parallel \sim t^{0.65}$ for large times.

Regarding the magnetic suppression of triple correlations, Schumann (1976) found in his numerical (DNS) experiments, that the magnetic field reduces both the angular energy transfer rate, and the nonlinear intercomponent energy transfer in the Reynolds stress equations (slow pressure–strain rate interaction). One can argue that also viscous dissipation is affected, because nonlinear interaction is responsible for the transfer of energy from larger to smaller scales; Sreenivasan and Alboussière (2000) showed that the k^{-3}-spectra observed in many MHD turbulence experiments are consistent with reduced energy transfer to the smaller dissipative scales.

Alemany *et al.* (1979) suggested that the nonlinear energy transfer rate in a magnetic field is reduced by a factor $(1+N)$, with N given by (15). The nonlinear energy transfer rate is usually estimated as ε/K. Using Alemany's suggestion, the reduced transfer rate is then $f_N\varepsilon/K$, with

$$f_N = \frac{1}{1+N} = \frac{\alpha}{\alpha+N^*}. \tag{18}$$

In the nonlinear phase of decay, we expect N^* to be constant and of order unity, so that $f_N \sim \alpha$ for small α. This means that the return-to-isotropy term π_α gets the desired properties for small α, if we use the *reduced* energy transfer rate in (10),

$$\pi_\alpha = C_{\alpha 2} f_N \frac{\varepsilon}{K}\left(\frac{1}{3}-\alpha\right). \tag{19}$$

If we assume the same reduction is appropriate for the viscous dissipation terms, the viscous dissipation in the K equation would be replaced by a model term $\varepsilon_K = f_N\varepsilon$, and the destruction term in the ε equation becomes $\varepsilon_\varepsilon = C_{\varepsilon 2} f_N \varepsilon^2/K$. Note that the closure variable ε would no longer directly represent the viscous dissipation of turbulent kinetic energy, but it would retain its role for estimates of time and length scales (K/ε and $K^{3/2}/\varepsilon$, respectively) of the energy-containing eddies.

A preliminary analysis of the properties of the modified MHD closure shows that the reduced viscous terms are negligible for large N, and the nonlinear energy decay is then given by $K \sim t^{-1/(C_{\varepsilon\alpha}-1)}$. If $C_{\varepsilon\alpha} = 1.5$, as proposed earlier, this gives $K \sim t^{-2}$. A slight adjustment to $C_{\varepsilon\alpha} = 1.59$ yields the result of Alemany *et al.*, $K \sim t^{-1.7}$, and the same length scale evolution as before.

Ongoing work aims at a better understanding of nonlinear effects, and modeling of them in a way that addresses the change of dynamics close to the 2D limit, as well as the expected magnetic suppression of triple correlations.

COMPARISON WITH EXPERIMENTS

The mercury experiments of Alemany *et al.* (1979) studied the decay of turbulence downstream of a grid, with a magnetic field oriented in the stream-wise direction. The magnetic interaction parameter was of order one. Hot film probes were used to measure the r.m.s. velocity parallel with the magnetic field, $u_\|$, and the parallel integral scale, $l_\|$ (from the one-dimensional spectrum). The results were presented using a non-dimensional time scale $Z - Z_0 = tU/M$, where U is the mean velocity, $M = 2$ cm is the grid mesh size, $Z_0 = 4$ is the location of the effective origin, and t is time.

For comparing model predictions with experiments, we here use the piecewise linear model (17) for the return-to-isotropy term π_α; the alternative model (19) gives similar results for the graphs shown here. Initial conditions for the simulation were computed from measured data in $Z-Z_0 = 2$. An ODE solver could then be run both forwards and backwards in time, in order to assure a sensible model behavior near the effective origin ($Z - Z_0 = 0$). Of the turbulence closure variables, only R_{33} can be compared directly with the measured $u_\|^2$. The initial condition for K was chosen to obtain isotropic turbulence ($R_{33} \approx R_{11}$) in the virtual origin. To compute initial conditions for ε and α, we used the scale relations introduced earlier,

$$\varepsilon = A\frac{K^{3/2}}{L}, \tag{20}$$

$$\alpha = \frac{1}{3}\left(\frac{L}{L_\|}\right)^2, \tag{21}$$

where A is presumably a constant of order unity. Here the initial parallel length scale $L_\|$ was taken from the measured $l_\|$ at $Z - Z_0 = 2$. The initial integral length scale L was then chosen so that backward integration yields $L \approx L_\|$ in the virtual origin, and the constant A was adjusted to match the time scales of experiment and simulation. The simulation was made with $A = 0.22$, and coefficients $\{C_{\varepsilon 2}, C_{\alpha 1}, C_{\varepsilon\alpha}, C'_{\alpha 2}, C_{\alpha 2}\} = \{1.92, 2, 1.5, 4.72, 0.4\}$.

The inverse of relations (20) and (21) can be used to present the simulation results in terms of the length scales L and $L_\|$. Figure 4 shows the decay of the field-parallel Reynolds stress, and the evolution of length scales, for the case of $U = 20$ cm/s and $B = 0.25$ T. Predictions with a standard K-ε model and experimental data for $B = 0$ are shown for comparison.

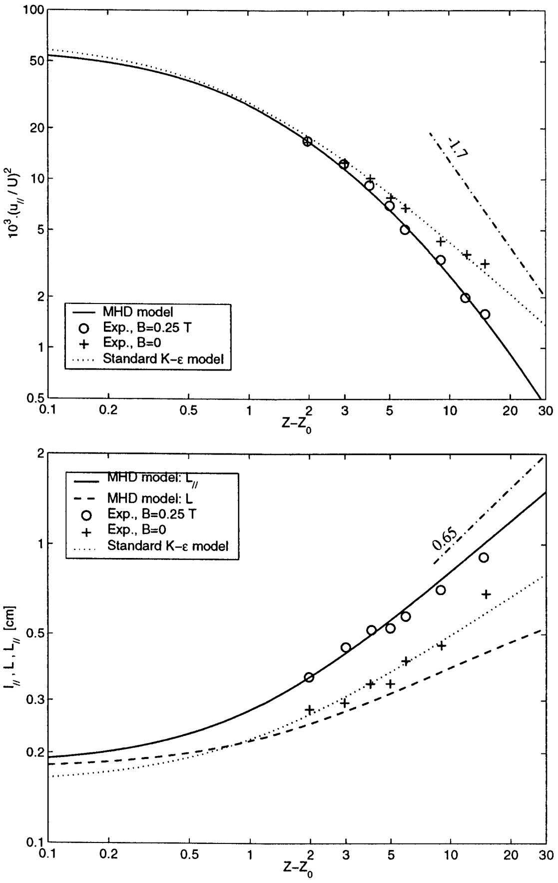

Figure 4: Comparison of model predictions with experimental data by Alemany *et al.* (1979) (U = 20 cm/s, B = 0.25 T). Decay of field-parallel Reynolds stress (above), and evolution of length scales (below). The horizontal axis shows normalized time (or normalized distance) from the virtual origin. Predictions with a standard K-ε model and experimental data for B = 0 are shown for comparison. The dash-dotted lines represent the asymptotic power-law behavior of the closure.

The Reynolds stress anisotropies are relatively small, so there is little difference between the evolution of K, and the individual stress components (not shown here). Compared with the non-magnetic case, the predicted integral length scale L evolves more slowly in the presence of a magnetic field, while the parallel scale $L_{\|}$ grows faster.

CONCLUSIONS

In this paper, we have tried to demonstrate the benefits of including structure and length scale information in closures for magnetohydrodynamic turbulence. For homogeneous turbulence, the proposed Reynolds stress closure is consistent with theory and available experiments for all values of the magnetic interaction parameter.

In future work, direct numerical simulations (DNS) can be of great value for understanding the various nonlinear mechanisms involved. For accurate predictions in engineering applications, another large effort will be to include effects of walls and inhomogeneities in the models. In contrast to homogeneous turbulence, near-wall turbulence in wall bounded MHD flows tends to become not only two-*dimensional* (long structures in the magnetic field direction), but also two-*component*, when the field-parallel stress component of growing structures begin to experience damping by nearby walls.

REFERENCES

Alemany, A., Moreau, R., Sulem, P. L., and Frisch, U., 1979, "Influence of an external magnetic field on homogeneous MHD turbulence", *J. de Mécanique*, Vol. 28, pp. 277–313.

Davidson, P. A., 1997, "The role of angular momentum in the magnetic damping of turbulence", *J. Fluid Mech.*, Vol. 336, pp. 123–150.

Lehnert, B., 1955, "The decay of magneto-turbulence in the presence of a magnetic field and Coriolis force", *Quart. Appl. Math.*, Vol. 12, pp. 321–341.

Moffatt, H. K., 1967, "On the suppression of turbulence by a uniform magnetic field", *J. Fluid Mech.*, Vol. 28, pp. 571–592, 1967.

Reynolds, W. C., 1989, "Effects of rotation on homogeneous turbulence", In *Proc. 10th Australasian Fluid Mech. Conf.*, Univ. of Melbourne, Australia.

Schumann, U., 1976, "Numerical simulation of transition from three- to two-dimensional turbulence under a magnetic field", *J. Fluid Mech.*, Vol. 74, pp. 31–58.

Sreenivasan, B., and Alboussière, T., 2000, "Evolution of a vortex in a magnetic field", *Eur. J. Mech. B - Fluids*, Vol. 19, pp. 403–421.

Widlund, O., Zahrai, S., and Bark, F. H., 1998, "Development of a Reynolds stress closure for modeling of homogeneous MHD turbulence", *Phys. Fluids*, Vol. 10, pp. 1987–1996.

Widlund, O., Zahrai, S., and Bark, F. H., 2000, "Structure information in rapid distortion analysis and one-point modeling of axisymmetric magnetohydrodynamic turbulence", *Phys. Fluids*, Vol. 12, pp. 2609–2620.