MODELLING EFFECTS OF FLOW CURVATURE AND FRAME ROTATION ON SHEARED TURBULENCE

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ABSTRACT*

A simple but accurate model of the effects of plane flow curvature and frame rotation on the development of sheared homogeneous turbulence is described. Model results are presented for the development of Reynolds Stresses and streamwise integral length scales under conditions of prolonged curvature, reversing curvature and a combination of curvature and streamwise acceleration. Comparison of model predictions to experimental data shows good agreement.

INTRODUCTION

Streamwise curvature, acceleration and rotation of plane shear layers commonly arise in technological flows such as the boundary layer along a wing or within the flow passages of turbo-machinery. Each of these mean flow characteristics has been studied by experimental and computational methods and found to have a dramatic effect on the scales and structure of the flow turbulence. As a result, modeling these measured effects has been given considerable effort and many schemes have been proposed which have had varying degrees of success. The model presented in this paper is a further attempt towards this goal but is limited to the simple case of homogeneous shear flow. This avoids the complications introduced by wall effects, entrainment or other forms of turbulence inhomogeneity but does prevent direct application to the practical flows described above. Nevertheless, the present model development may provide insights into the basic mechanisms involved that can be used to improve existing, more robust models.

A curved uniform shear flow is illustrated in Figure 1. In the traditional description of this flow, using curvilinear (s,n) coordinates, the strength of curvature relative to the shear is characterized by the dimensionless parameter $S=(U/R)/(dU/dn)$. For flows with $S>0$, a convex wall flow, the turbulence is stabilized; whereas flows with $S<0$, a concave wall flow, the turbulence is destabilized. To facilitate formulation of the model, two additional coordinates are introduced. One is a fixed inertial system, X^*_i, and the other, X_i, rotates at the rate of the mean shear, Ω (=U/R), so that X_1 is locally tangent to the mean streamline. The mean shear rate in the rotating shear frame is $dU_1/dx_2 = dU/dn - U/R$.

Rotating uniform shear flow is illustrated in Figure 2. The inertial frame remains fixed and the shear frame rotates at the rate Ω. The strength of the rotation, relative to the shear, is $\Omega/(dU_1/dx_2)$. Analogous to curvature, mild rotation which reinforces the rotation of the mean shear stabilizes the turbulence and vice-versa. Curved and rotating uniform shear flows are equated according to $\Omega/(dU_1/dx_2) = S/(1-S)$.

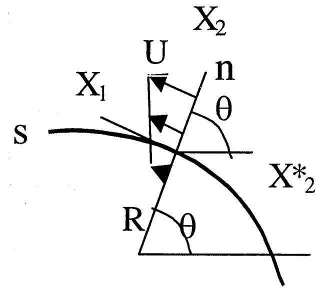

Figure 1: Curved uniform shear flow showing inertial and shear frame coordinates along with the traditional curvilinear coordinates.

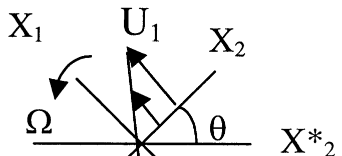

Figure 2: Rotating uniform shear flow showing inertial and shear frame coordinates.

The present model is based on a previously published geometric explanation of weak curvature effects [1]. The primary assumptions of that analysis were: 1) the mean shear produces turbulence at each position s having a fixed structure typical of uncurved uniform shear flow when referenced to the

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shear frame, 2) the turbulence eddies, once produced, are swept along by the stream unchanged when referenced to the inertial frame, and 3) the shear determines the time scale for adjustment of the energy containing motions. The effects of curvature on the turbulence arise from the coordinate axes turning away from the existing eddies of the flow while the rotating mean shear produces new eddies with continuously varying orientations. The stresses at a position s are calculated as the accumulated effect of all the eddies at that position; each having a different orientation which depends on its site of production. A limited model developed in [1] that was based on these premises, and the assumption of constant turbulence kinetic energy, was found to give accurate predictions of the measured values of the normalized Reynolds stress anisotropy in uniformly sheared turbulence subjected to prolonged constant curvature.

Holloway [2] assumed the above arguments apply to all aspects of the turbulence structure and extended the model equations to the two-point velocity covariance. This allowed prediction of the effects of curvature on the ratios of the streamwise integral length scales of the flow.

The present paper describes a simpler and yet more complete model than that described above. The changes that have been made are: 1) the integral equations describing the development of the anisotropy in [1,2] have been reformulated as differential equations, and 2) the assumptions of constant turbulence kinetic energy and length scale have been replaced by equations which govern the development of these scales. The result is a model capable of predicting the Reynolds stresses and the streamwise integral length scales of uniform shear flow subjected to plane curvature. The model naturally applies to rotating uniform shear flow without modification although its accuracy in such flows has not yet been tested directly. In addition, the model is extended to the effects of acceleration by including the effects of the associated streamwise strain on the mean shear rate and direction.

MODEL DESCRIPTION

Anisotropy

The anisotropy of the two-point velocity covariance is defined as

$$M_{ij}(\vec{r}/L,t) = \left(\overline{u_i u_j(\vec{r}/L,t)} - I_{ij}(\vec{r}/L,t)\right)/q^2 \quad \textbf{(1)}$$

where the scales used for normalizing the velocity covariance and spatial separation, $\vec{r}$, are calculated from

$$q^2(t) = \overline{u_i u_i}(0,t) \qquad \textbf{(2)}$$

$$L(t) = \frac{1}{q^2}\int_0^\infty \overline{u_i u_i}(\vec{r},t)dr \cdot$$

The anisotropy and scales are shown as functions of the time, t, in the mean convected frame. $I_{ij}(\vec{r},t)$ is the two-point covariance for an isotropic turbulence having the same q^2 and L as the uniform shear flow under consideration.

The development of the anisotropy tensor in the inertial frame is modelled using the simple equation

$$\tau\frac{dM^*_{ij}}{dt} + M^*_{ij} = \left(M_{kl}\right)_{(r)} e_{ik} e_{jl} \qquad i,j = 1,2 \qquad \textbf{(3)}$$

where the time constant, τ, is taken to be simply proportional to the shear rate but increasing with the spatial separation distance $\vec{r}/L$. The subscript (r) in equation (2) denotes the reference values of anisotropy that occur in rectilinear uniformly sheared turbulence, i.e. no curvature or rotation.The out of plane anisotropy component was calculated using M^*_{33} = - M^*_{11} – M^*_{22}. Once M^*_{ij} has been determined from the integration of Equation (2), the anisotropy in the frame of the shear may be determined by rotation of the components from the inertial frame using

$$M_{ij} = M^*_{kl} e_{ki} e_{lj} \qquad \textbf{(4)}$$

The effects of flow curvature and rotation are introduced by way of the rotational transformation, e_{ij}, which relates the coordinates of the inertial and shear frames. The angle between these axes in plane flow is calculated from

$$\theta = \int_0^t \Omega dt^* \qquad \textbf{(5)}$$

where Ω (=U/R) is the rate of shear frame rotation.

The Reynolds stress anisotropy, $m_{kl}(t) = M_{kl}(0,t)$, and an equation for its development is derived from equation (3) with $\vec{r}=0$. Reference values for the Reynolds stress anisotropy vary slightly among uniformly sheared flows with the typical values being [1]

$$(m_{kl})_{(r)} = \begin{bmatrix} .17\alpha & -.14 & 0 \\ -.14 & -.14\sqrt{\alpha} & 0 \\ 0 & 0 & -.03 \end{bmatrix} \qquad \textbf{(6)}$$

where the flow is in the X_1 direction and the shear in the positive X_2 direction. Holloway and Tavoularis [1] did not include α in their reference values and obtained good estimates of the curvature effects on the shear component of the stress anisotropy. Here we will take $\alpha = m_{12}/(m_{12})_r$ as an arbitrary measure to improve the prediction of the normal stresses under stabilized conditions. The value of τ for zero separation provided in [1], $\tau_o = 1.5(dU_1/dx_2)$, will be used for all calculations.

The dimensionless stresses can be calculated from the anisotropy by using the following

$$\frac{\overline{u_1^2}}{q^2} = m_{11} + \frac{1}{3} \qquad \frac{\overline{u_2^2}}{q^2} = m_{22} + \frac{1}{3} \qquad \textbf{(7)}$$

$$\frac{\overline{u_3^2}}{q^2} = m_{33} + \frac{1}{3} \qquad \frac{\overline{u_1 u_2}}{q^2} = m_{12}$$

The length scale anisotropy is defined as the integral of M_{ij} with respect to the distance between points for a given direction of the displacement vector,

$$\Lambda_{ij}(\frac{\vec{r}}{r},t) = \int_0^{\infty} M_{ij}(\vec{r}/L,t)\,d(r/L) \quad \textbf{(8)}$$

An equation for Λ^*_{ij}, which is relative to the inertial frame, may be formed from the integration of Equation (3) keeping the direction of $\vec{r}$ fixed. The result is

$$\bar{\tau}\frac{d\Lambda^*_{ij}}{dt} + \Lambda^*_{ij} = (\Lambda_{kl})_{(r)} e_{ik} e_{jl} \quad i,j = 1,2 \qquad \textbf{(9)}$$

where $\bar{\tau}$ is an average value of the time constant appearing in equation (3). The out of plane normal component is calculated as $\Lambda^*_{33} = -\Lambda^*_{11} - \Lambda^*_{22}$. The integral length scale anisotropy in the shear frame is subsequently calculated by the rotational transformation

$$\Lambda_{ij} = \Lambda^*_{kl} e_{ki} e_{lj} \qquad \textbf{(10)}$$

Equation (9) is simple, but its use is complicated by the fact that reference values of the integral length scale anisotropy, $(\Lambda_{ij})_r$, must be provided for a range of displacement vector directions. This arises because the local shear frame in a rotating or curved flow changes its orientation relative to a fixed $\vec{r}$. For example, the development of Λ^*_{ij} for a streamwise integral length scale anisotropy would depend on reference values for non-streamwise separations at upstream positions. To simplify the present calculations, the changes in orientation of the displacement vector relative to the shear frame will be ignored and the integral length scale anisotropy for streamwise displacements will be calculated solely using the reference values for streamwise displacement. The error associated with this approximation increases with the rate of curvature or rotation relative to the time constant τ^{-1}. For more details concerning this approximation see reference [2]. Typical reference values of $(\Lambda_{ij})_r$ for streamwise separations are [2]

$$(\Lambda_{kl})_{(r)} = \begin{bmatrix} 0.25\alpha & -0.19 & 0 \\ -0.19 & -0.16\sqrt{\alpha} & 0 \\ 0 & 0 & -0.09 \end{bmatrix} \qquad \textbf{(11)}$$

The value, $\bar{\tau} = 2.25(dU_1/dx_2)^{-1}$, was used in all the present calculations.

The dimensionless streamwise length scales, L_{ij}/L may be calculated using m_{ij} and Λ_{ij} as for example

$$\frac{L_{11}}{L} = \frac{\Lambda_{11} + \frac{1}{2}}{m_{11} + \frac{1}{3}} \qquad \frac{L_{22}}{L} = \frac{\Lambda_{22} + \frac{1}{4}}{m_{22} + \frac{1}{3}} \qquad \textbf{(12)}$$

$$\frac{L_{33}}{L} = \frac{\Lambda_{33} + \frac{1}{4}}{m_{33} + \frac{1}{3}} \qquad \frac{L_{12}}{L} = \frac{\Lambda_{12}}{m_{12}}$$

Scales

This section describes equations for the development of q^2 and L which may be combined with the equations of previous section to allow predictions of the Reynolds stress and streamwise integral lengths

The exact equation for q^2 in homogeneous shear flow may be written [3]

$$\frac{dq^2(t)}{dt} = -2\overline{u_1u_2}(t)\frac{dU_1}{dx_2} - \varepsilon_{q^2} \qquad \textbf{(13)}$$

where the last term on the right hand side is the dissipation rate of the turbulence kinetic energy. In the present work the dissipation rate was modelled as being proportional to the kinetic energy (delayed by an interval equal to the adjustment time of the Reynolds stress anisotropy) and the mean shear

$$\varepsilon_{q^2} = -\frac{q^2(t-\tau_o)}{3\tau_o} \qquad \textbf{(14)}$$

Equation (14) was found to work well for the present shear dominated flows and is a novel alternative to the standard k-ε formulation. Obviously it would not be applicable to unsheared flows such as decaying grid turbulence. The effects of curvature or rotation in equation (13) arise primarily from their effects on the turbulence shear stress and mean shear rate. The factor of 3 was chosen to give the correct exponent of growth for uncurved uniform shear flow.

An exact equation for the development of q^2L in homogeneous shear flow, from which L can be obtained, can be formulated from the the two-point velocity covariance equation [3] by contraction and integration with respect to streamwise separation distance

$$\frac{dq^2L(t)}{dt} = -2\overline{u_1u_2(t)}L_{12}(t)\frac{dU_1}{dx_2} - \varepsilon_{q^2L} \qquad \textbf{(15)}$$

Equation (15) requires modelling of the production term, because it involves the exact integral length scale, and a dissipation rate. Both models were chosen simply on the basis of providing the correct growth rate for the length scale in uniformly sheared turbulence. Under such conditions both q^2 and L grow exponentially with exponents in a 2:1 ratio since L is related to the speed, q, by the shear rate. This condition can be met under asymptotic conditions if

$$L_{12} = 1.5L \quad \text{and} \quad \varepsilon_{q^2} = \frac{q^2L\left(t-\frac{2}{3}\tau_o\right)}{2\tau_o} \qquad \textbf{(16)}$$

The validity of equations (15) and (16) for curved flows will be evaluated in the next section.

COMPARISON TO EXPERIMENTAL DATA.

Model predictions will be evaluated by direct comparison to a selection of experimental data from uniformly sheared turbulence subjected to constant curvature, reversing curvature, and constant curvature with flow acceleration. In each experiment the uniform shear flow was developed in a straight tunnel until it had a self-preserving structure where the stress anisotropy approaches constant values and the velocity and length scales of the turbulence grow exponentially. The flows were then guided tangentially into a curved tunnel of the appropriate shape where the linearity of the mean shear and uniformity of the turbulence statistics was approximately preserved in the central core of the tunnel. Measurements for the development of the turbulence under the influence of curvature were made along the tunnel centerline. All values reported are expressed in the shear frame coordinates as described in Figure 1.

The components of Reynolds stresses and integral length scales measured near the end of the straight tunnel served as initial conditions for the calculations and were used to specify reference values of the anisotropy for each flow. The streamwise integral length scales were estimated by integrating the temporal autocorrelations up to the first zero and applying Taylor's frozen flow hypothesis.

Prolonged Curvature.

The problem of uniformly sheared turbulence subjected to prolonged constant curvature has been studied by Holloway and Tavoularis [4] using a wide range of experimental conditions that included constant curvature tunnels of different radii and different mean shear rates. They found that the turbulence structure of these flows was strongly affected by curvature and that these effects could be correlated using the curvature parameter S. In this paper the model is applied to the shear dominated flows in which $-0.2<S<0.2$.

Figure 3 shows measured and predicted values for the exponents of growth for q^2 and L^2 defined as $\kappa_q{}^2 = (dq^2/dt)/q^2$ and $2\kappa_L = (dL^2/dt)/L^2$. The data shows that flow curvature has a strong effect on these exponents, but less effect on their ratio. The model tends to underpredict the effects of curvature on the exponents but does give the correct trend over a wide range of S.

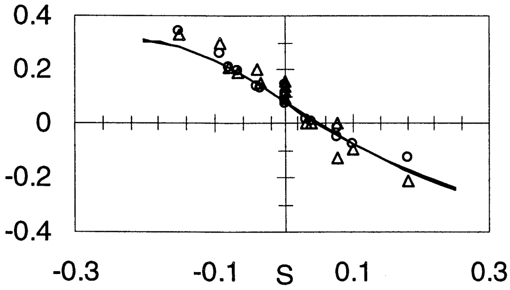


Figure 3: Exponents of growth of q^2 (O) and L^2 (△) measured in uniform shear flow subjected to prolonged constant curvature of various strengths [4]. The lines represent model results.

Figure 4 shows a comparison of the measured and predicted values of m_{ij} which develop asymptotically. The dashed lines show the predictions for $\alpha=1$ and the solid lines the predictions using $\alpha = m_{12}/(m_{12})_r$. The values of m_{ij} are fixed at S=0 by the choice of its reference values (equations (6) and (11)). The model accurately predicts the redistribution of energy among the component directions in the shear frame and the decline in anisotropy with increasingly positive values of S.

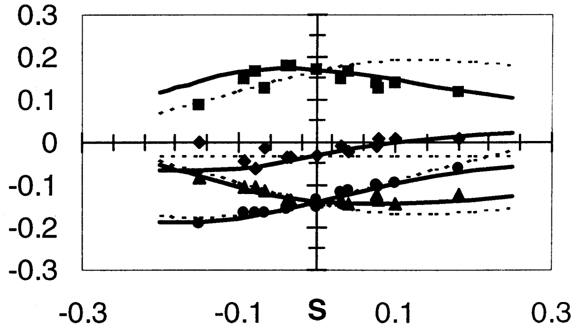


Figure 4: Values of the Reynolds stress anisotropy, m_{ij} , measured in uniform shear flow subjected to prolonged constant curvatures of various strengths. Symbols representing the data [4] are for: (i=1, j=1) ■; (1,2) ●; (2,2) ▲; (3,3) ♦. The solid lines represent model calculations.

Figure 5 shows a comparison between the measured and predicted values of the streamwise length scale anistropies. The alternative definitions; L_{uv}/L_{uu}, $2L_{vv}/L_{uu}$-1, and $2L_{ww}/L_{uu}$-1 are used to allow an easier comparison to published experimental data. The trends of the data are accurately predicted for the normal components but less so for the shear component. For example: $2L_{vv}/L_{uu}$-1 correctly shows a minimum at S~0.05 and that $2L_{ww}/L_{uu}$-1 is negative up to small values of positive S and then increases to positive values for S > 0.1. The predictions of L_{uv}/L_{uu} decreases with increasingly positive S but at too slow a rate compared to the data. This error in L_{uv} would prevent an accurate calculation of the production of q^2L in equation (15). The loss of anisotropy among the length scales for positive S is similar to that observed for the stress.

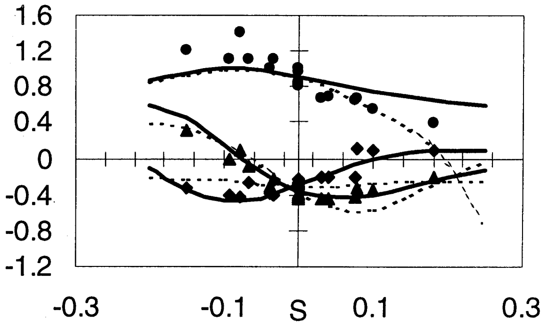


Figure 5: Components of streamwise integral length scale anisotropy measured in uniform shear flow subjected to prolonged constant curvatures of various strengths. Symbols as in Figure 4.

Reversing Curvature.

A schematic of the uniform shear flow with reversing curvature studied by Chebbi et al. [5] is shown in Figure 6. Figure 7 shows the streamwise variation of mean shear rate and rate of rotation of the shear frame which were derived from measurements [5] and used in the model calculations. In particular, considerable effort was made to match the mean streamline shape reported by Chebbi [6] for this flow. This study showed that the model predictions were very sensitive to small adjustments in mean streamline shape.

The turbulence enters the curved tunnel at s=0 and for the first metre is subjected to stabilizing curvature with S ~ 0.06, for the second metre to destabilizing curvature with S ~ -0.06 and the final metre of the tunnel is uncurved. As a result the anisotropy and rates of growth of the turbulence tends to adjust between the asymptotic states of Figures 3, 4 and 5.

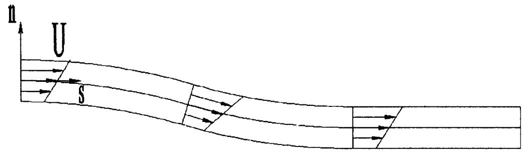


Figure 6: Schematic of the uniform shear flow with reversing curvature studied by Chebbi et al. [5,6]. Streamwise distance along the tunnel centreline is measured in metres from the start of curvature.

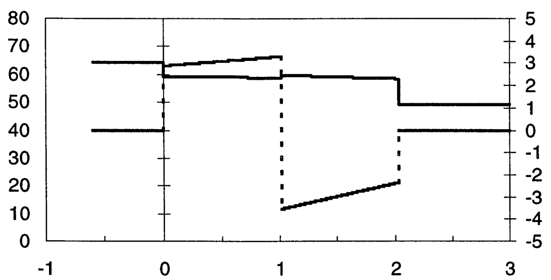
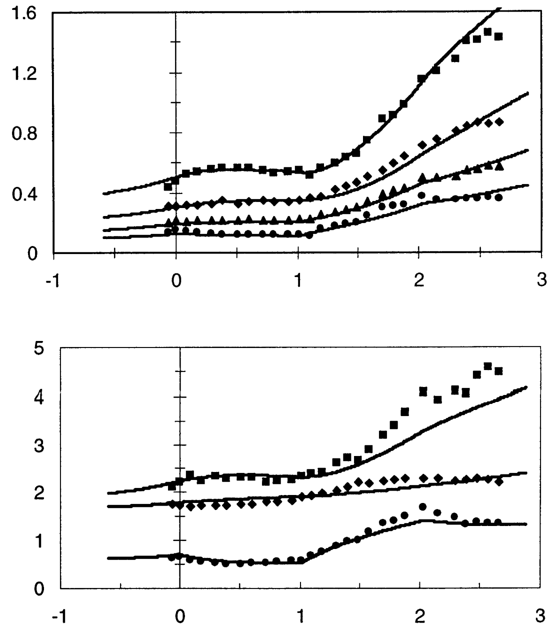


Figure 7: Streamwise variation of the mean flow parameters; (dU/dn–U/R) [s^{-1}] and U/R [s^{-1}], along the centreline of uniform shear flow with reversing curvature. The lines represent fits to experimental data of [5,6]. The dashed line refers to U/R and the right ordinate.

Figures 8 and 9 compare the model predictions of Reynolds stresses and the streamwise integral length scales to measured values. These show very strong agreement in the curved sections but less so in the straight section where curvature is removed.



Figures 8, 9: Development of Reynolds stress components and streamwise integral length scales for uniform shear flow with reversing curvature. Symbols representing the data of [5, 6] are: $\overline{u_1^2}$ and L_{11}, ■; $\overline{u_1u_2}$, ●; $\overline{u_2^2}$ and L_{22}, ▲,; $\overline{u_3^2}$ and L_{33} ♦. Lines represent model calculation using the mean flow conditions of Figure 7.

Combination of curvature and acceleration.

A schematic of one of several uniform shear flows combining constant curvature and flow acceleration studied by Roach [7] is shown in Figure 10. The curvature of the tunnel centreline is constant over its length and the acceleration is applied in the first half of the tunnel by plane convergence of the upper and lower walls.

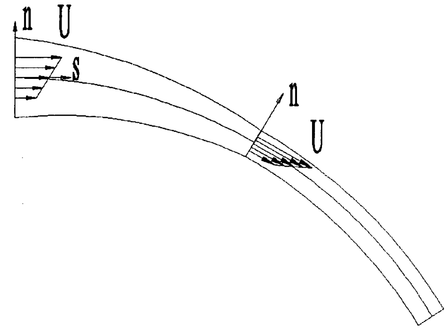


Figure 10: Schematic of uniform shear flow subjected to constant flow curvature and streamwise acceleration [7].

The mean acceleration of the flow is accounted for in the model calculations by including the effects of the associated mean streamwise rate of strain. This additional strain rate effectively increases the mean shear rate and rotates the direction of maximum mean shear towards the streamwise direction. The modified mean shear rate is

$$\left(\frac{dU_1}{dx_2}\right)^+ = \sqrt{\left(\frac{\partial U}{\partial n}-\frac{U}{r}\right)^2+\left(2\frac{\partial U}{\partial s}\right)^2} \qquad \textbf{(17)}$$

and the angle of rotation of the maximum mean shear towards the streamwise direction is

$$\beta = \frac{1}{2}\arctan\left(\frac{2\partial U/\partial s}{\partial U/\partial n - U/R}\right) \qquad \textbf{(18)}$$

The angle β is subtracted from the angle θ used in e_{ij} of equation (3) and (9) but not subtracted from the angle θ used in e_{ij} of equations (4) and (10). This apparent inconsistency arises because the mean shear is rotated by β but the coordinate axes used for presentation of the results are not. Figure 11 shows the streamwise variation of, U/R, $(dU/dn)^+$, and β which were derived from measurements [7] and used in model calculations. The turbulence enters the curved tunnel at s=0 and is subjected to a combination of stabilizing curvature with on average S~0.16 and increasing streamwise strain rate during the first 1.5 metres of length. In the second half of the curved tunnel the turbulence is subjected to curvature alone but with substantially higher velocity giving S~0.35 on average. The streamwise strain increases the mean shear rate by 50% and rotates it by up to 30°. In the present case of stabilizing curvature this additional rotation opposes the rotation due to curvature. The streamwise strain rate also has a negative effect on the production terms of equations (13) and (15).

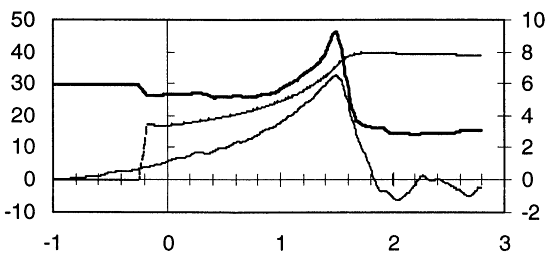


Figure 11: Streamwise variation of the mean flow parameters; (dU/dn–U/R) [s^{-1}, heavy solid line] , β [degrees, light solid line], and U/R [s^{-1}, dashed line] measured along the tunnel centreline of the uniform shear flow with constant curvature and streamwise acceleration. U/R is referred to the right ordinate.

The measured and predicted development of the Reynolds stresses and streamwise integral length scales are shown in Figures 12 and 13. Over all it is clear that the kinetic energy and length scales of the turbulence decay under this combination of stabilizing curvature and acceleration at a faster rate than would occur for curvature alone. This is clearly evidenced by the development after the removal of acceleration. This is also true for the streamwise components of stress and length scale but not for the other components which decay at a reduced rate or increase in the accelerating portion of the tunnel. The model predictions do display the data trends but the abrupt change in decay rate of the stresses at the entrance to the curved tunnel is not predicted. This may be due to a distortion of the flow at the tunnel entrance that has not been accounted for or perhaps a shortcoming of the model.

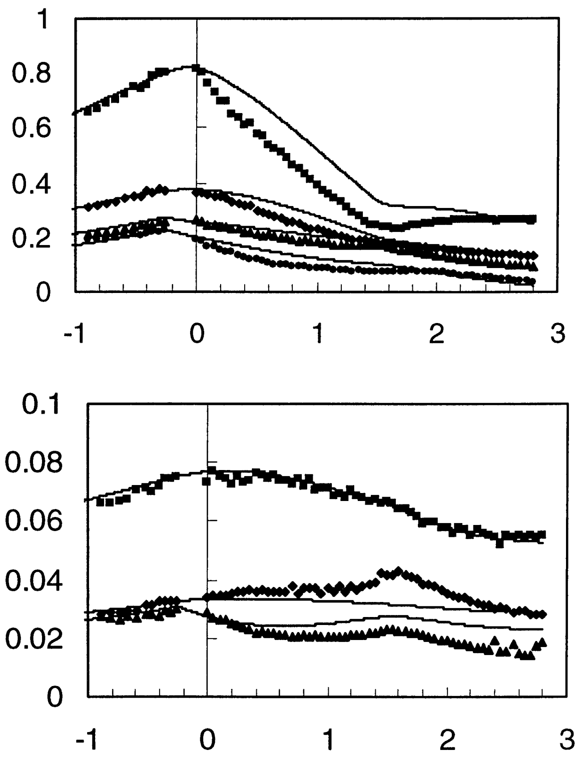


Figure 12,13: Development of Reynolds stress and streamwise integral length scales for uniform shear flow with reversing curvature. Measurements from [3]. Symbols as in Figures 8 and 9.

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