SOME PERSPECTIVES ON PRESSURE-STRAIN CORRELATION MODELING

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ABSTRACT

In complex turbulent flows, it is argued that the standard pressure-strain correlation model must be augmented by additional tensors to accurately account for complicating influences (such as system rotation, streamline curvature, buoyancy, etc.). This leads to the question, how many tensors do we need to *adequately* represent the pressure strain correlation. Representation theory indicates that a *complete* function-space representation of pressure-strain correlation in terms of Reynolds stress anisotropy, mean strain rate and mean rotation rate involves tens of basis tensors. On the other hand, purely from dimensional arguments it can be shown that at a given point in space, the pressure-strain correlation can be expressed completely in terms of only three basis tensors in two-dimensional mean flow and only five in three-dimensional flows. So is the number of basis tensors required thirty or is it five? In this paper, we will examine and explain the difference between the two numbers and how that impacts on modeling pressure-strain correlation or any other traceless symmetric second-order tensor in turbulence. Specifically we will compare merits of short and long tensor representations. We also derive mathematically equivalent shorter tensor representations of popular longer versions of pressure-strain correlation models.

INTRODUCTION

Modeling the rapid part of the pressure-strain correlation continues to be difficult especially in complex turbulent flows. In order to account for the complicating effects of extra rates of strain (system rotation, stream- line curvature, buoyancy etc.), flow geometry (wall-blockage effects) and non-locality (in space and time) researchers in recent time have added extra tensors to the standard representation of pressure-strain correlation. The coefficient of each new tensor is then 'calibrated' to yield acceptable performance in the flow of interest. Some models proposed recently consist of upto ten tensor terms for even two-dimensional mean flows. This practice leads to several important questions:

1. How does the addition of the new terms affect the model performance in bench mark flows? For example, if an added tensor is not linearly independent of tensors in the standard representation, that can have an adverse influence in bench-mark flows in which the standard model has been calibrated.

2. How is the model calibration performed? Are the coefficients of the standard model left unchanged or are they also recalibrated.

3. Is the set of model coefficients the most optimal? Are there other combinations of coefficients that would yield better performance.

4. How sensitive is the model performance to changes in the model coefficients? Can small changes in the model coefficients lead to drastically different model predictions.

It is preferable to keep the number of tensors in the pressure-strain correlation model down to a small number. This leads to the question 'what is the optimum number of tensors required for a complete representation of pressure-strain correlation?' Further, how will this number change with the addition of new phenomena influencing turbulence? These questions can be approached from two view points. The first is to appeal to representation theory for the complete integrity tensor basis. This typically leads to an unmanageably large tensor basis. The second approach is to use physical dimensionality arguments. It will

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be demonstrated that the absolute minimum number of tensors needed for a complete representation of pressure-strain correlation (or any other symmetric, traceless tensor) at any given point in space is three in two-dimensional turbulence and five in the three-dimensional case. Further, these numbers are independent of the influences that may complicate turbulence. The number is dependent only on the physical dimension in which the symmetric, traceless tensor resides. The main objective of this paper is to understand why the two approaches lead to completely disparate answers and suggest an adequate and manageable representation for the pressure strain correlation model.

PRESSURE-STRAIN CORRELATION MODELING

The 'so-called' standard pressure-strain correlation model has its origins in Launder, Reece and Rodi (1975) (LRR model) and it has the following form:

$$\begin{aligned}\phi_{ij} = &\ -(C_1^0\varepsilon + C_1^1 P)b_{ij} + C_2 K S_{ij} \\ &+C_3 K(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}) \\ &+C_4 K(b_{ik}W_{jk} + b_{jk}W_{ik}), \qquad (1)\end{aligned}$$

where the C's are model coefficients and

$$\begin{aligned}S_{ij} &= \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i});\ W_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}); \\ K &= \frac{1}{2}\overline{u_i u_i};\quad b_{ij} = \frac{\overline{u_i u_j}}{2K} - \frac{1}{3}\delta_{ij}. \qquad (2)\end{aligned}$$

In the LRR model, the coefficients are

$$C_1^0 = 3.0; C_1^1 = 0; C_2 = 0.8; C_3 = 1.75; C_4 = 1.31. \qquad (3)$$

A variant of the above model also given in LRR is

$$C_1^0 = 3.6; C_1^1 = 0; C_2 = 0.8; C_3 = 1.2; C_4 = 1.2. \qquad (4)$$

For the quasilinearized Speziale, Sarkar and Gatski (1991) (SSG) model, the coefficients are

$$C_1^0 = 3.4; C_1^1 = 1.8; C_2 = 0.36; C_3 = 1.25; C_4 = 0.4. \qquad (5)$$

This family of models has been quite successful in a variety of benchmark flows. However, in some important complex flows (eg., elliptic flows) it fails to capture even the qualitative trends correctly. To improve model performance in complex flows, many researchers have added extra tensor functions to the standard form. Representation theory is used to guide in the selection of new tensor functions.

REPRESENTATION THEORY AND INTEGRITY BASIS.

Representation theory, which has its roots in classical invariant theory, is a powerful analytical tool routinely used in continuum mechanics for developing constitutive relationships. More recently, representation theory has been used in turbulence modeling for determining the complete tensor basis. This *integrity basis* is the (smallest) set of irreducible tensor functions that completely span the function space.

If we postulate that the pressure-strain correlation depends on the symmetric and antisymmetric parts of the velocity gradient tensor, then the integrity basis consists of ten algebraically irreducible tensors:

$$\phi_{i,j} = \sum_{n=1}^{N} a_n I_{ij}^n \qquad (6)$$

where N is the number of basis tensors (ten in this case), a_n is the coefficient of the n-th basis tensor I_{ij}^n. The coefficient a_n is an unknown polynomial function of the scalar invariants of the basis tensors. If we believe that the pressure-strain correlation will also depend on other quantities (eg., b_{ij}), the number of the integrity basis tensors balloons to over 40. If we are also interested in buoyancy effects this number will go up further. There have been a few papers in literature that propose pressure-strain correlation models that include over twenty tensors. Such lengthy models are not computationally viable. Even more importantly, calibrating the coefficients is very difficult when the representation is large. Are all the integrity tensors really necessary for deriving practical models?

Although the integrity basis is irreducible in function space, they typically constitute a redundant basis in physical space. This is most easily understood by considering vectors rather than tensors. Let us postulate that a vector v_i is dependent on ten different vectors I_i^1 to I_i^{10}. The integrity basis approach will lead to a representation of the type

$$v_i = \sum_{n=1}^{n=10} a_n I_i^n + \cdots \qquad (7)$$

Consider any three linearly independent vectors from within $\mathbf{I}$. We will denote this subset by $\mathbf{M}$. Since v_i resides in a three-dimensional space, we can unequivocally state that the subset $\mathbf{M}$ forms a complete set of basis tensors:

$$v_i(\mathbf{x}, t) = \sum_{n=1}^{n=3} d_n M_i^n(\mathbf{x}, t) \qquad (8)$$

The coefficients have a simple interpretation: d_n is the projection of vector $\mathbf{v}$ on vector $\mathbf{M}^n$, i.e, $d_n = \mathbf{v} \cdot \mathbf{M}^n$. Infact, even the other vectors in $\mathbf{I}$ can also be expressed in terms of the basis $\mathbf{M}$. Clearly, of the ten integrity basis vectors only three can be linearly independent in physical space. Does that mean that the three-vector subset $\mathbf{M}$ is all that we require to completely represent $\mathbf{v}$? In regions of physical space where all the M_i^n vectors are non-degenerate (non-zero and linearly independent) the answer is clearly yes. But in regions of physical space where one or more of M_i^n goes to zero, the representation is incomplete. Then we will need a different subset of three non-zero independent vectors. The integrity basis can be considered the superset of all three-vector subsets required to completely represent the vector $\mathbf{v}$. Similar arguments are valid for tensors also.

Important inferences. If we want a complete representation that is valid in all situations, real or imagined, then we need the full integrity basis. But as seen before, carrying all the integrity basis tensors in the model representation is a great burden: both from the point of view of model calibration and application. In a broad class of problems, it is quite possible that the physics of the phenomenon does not permit the occurrence of many of the situations against which the integrity basis provides us with a safeguard. In those cases, the integrity basis can be be substantially reduced without compromising the validity of the representation.

In modeling the pressure-strain correlation, if we can identify a small number (equal to or more than that dictated by dimensionality) of tensors from the integrity basis that are non-degenerate in the domain of interest, then that subset can ably serve as a complete representation. In fact, this is the reason for the success of the standard model which does not include a large number of the integrity basis tensors in its representation. In order to maximize the applicability of the model, the choice of the subset must be made judiciously. In this paper, we will call such a subset the *optimal basis.*

A caveat. The coefficients in the integrity representation are (unknown) polynomials of the invariants of the tensors. The functional forms of the coefficients in the 'optimal basis' are completely unknown. However, this is not too disadvantageous. Irrespective of which method is used, the coefficients will be determined purely empirically. The fact that the 'optimal basis' coefficients are of unknown functional form is therefore irrelevant for turbulence applications. If a formal approach were to be available to determine the polynomial coefficients of integrity basis, then we should re-evaluate this stance.

SHORTEST REPRESENTATION FOR PRESSURE-STRAIN CORRELATION

Jongen and Gatski (1998) demonstrate that the Reynolds stress anisotropy tensor can be completely represented at any point in physical space in terms of only five tensors. The reasoning is quite simple. The anisotropy tensor is symmetric and traceless and resides in a three dimensional physical space. As a result, it has only five linearly independent components requiring five tensors only for its complete description. Certainly, the same must be true for pressure-strain correlation or any other symmetric and traceless tensor in turbulence.

The 'optimal basis' for pressure-strain correlation representation can be as small as three tensors in two-dimensional mean flow. For these flows, the standard representation that involves four tensors is completely adequate unless any one of b_{ij}, S_{ij} or W_{ij} become zero. Otherwise, this model form can capture the effects of buoyancy, wall-blockage, rotation, streamline curvature, ellipticity, history and non-locality, etc., without the addition of any further tensor terms. The effect of these other influences will manifest through the coefficients which will be functions of the scalar invariants of the different phenomena. An immediate implication is that we do not need the quadratic anisotropy tensor at all in the rapid pressure-model. The second scalar invariant of course can appear in the coefficients. The standard form with invariants from these tensors can equally perform the task. Of course, for three dimensional mean flows we will need a few more basis tensors in the optimal representation but not many.

One example that demonstrates the wide range of applicability of the standard model form involves the elliptic flow (Blaisdell and Sheriff, 1996). The poor performance of LRR and SSG model in this flow led to the speculation that perhaps spin and structure function tensors need to be included in the pressure-strain correlation model (Kassinos

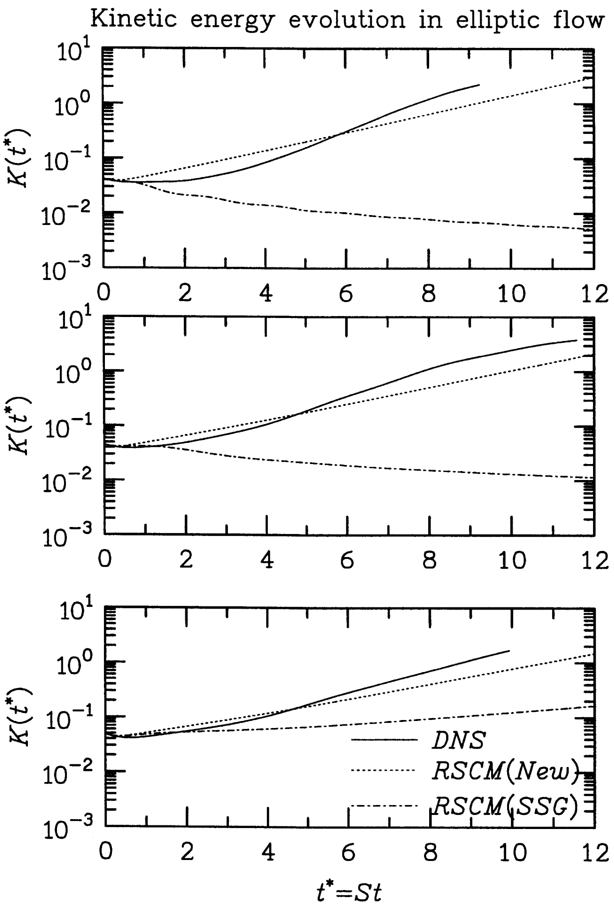

Figure 1: Evolution of kinetic energy in elliptic flows. DNS data from Blaisdell and Sharif (1996). The figures from top to bottom correspond to ellipticity factors of 1.5, 2.0 and 3.0. These correspond to $\eta_1 = 0.13, 0.26 and 0.39$.

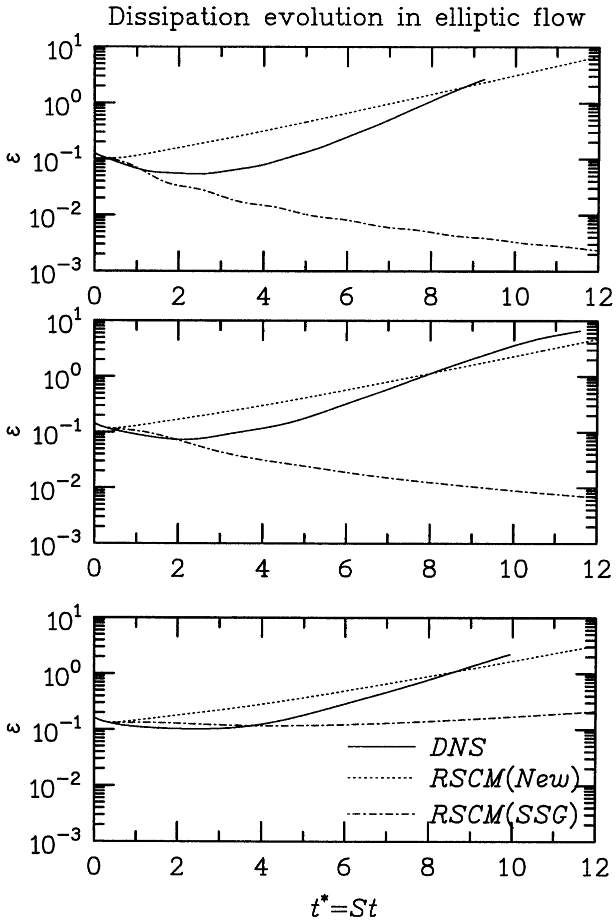

Figure 2: Evolution of dissipation in elliptic flows. DNS data from Blaisdell and Sharif (1996).

and Reynolds, 1994). However, it was shown in Girimaji (2000) that the standard form of the model is quite adequate for this purpose, provided the coefficients are made sensitive to the appropriate invariants. In Figure 1, the evolution of kinetic energy in an elliptic streamline flow is shown as a function of time for three different ellipticity values. The direct numerical simulation (DNS) result is shown with the solid line. For all ellipticity values, DNS shows that the kinetic energy ultimately increases. The SSG model completely misses this trend. For low ellipticity values this model indicates energy decay. The behavior of LRR model is only marginally better (not shown). The Girimaji (2000) model (also of standard form) does quite well in capturing the DNS behavior. The comparison of the dissipation evolution is made in Figure 2.

This modeling approach shifts the emphasis from various tensors to modeling their invariants. In a two-dimensional mean flow consider the following representation:

$$\phi_{ij} = H_1 S_{ij} + H_2(S_{ik}W_{kj} - W_{ik}S_{kj}) + H_3(S_{ik}S_{kj} - \frac{1}{3}\delta_{ij}S_{lm}S_{lm}) \quad (9)$$

It can be shown (see further below) that the coefficients H are projections of the pressure-strain tensor on each of the basis tensors:

$$\begin{aligned} H_1 &= \phi_{ij}S_{ij}/\eta_1 \\ H_2 &= \phi_{ij}(S_{ik}W_{kj} - W_{ik}S_{kj})/(2\eta_1\eta_2) \\ H_3 &= 6\phi_{ij}S_{ij}S_{jk}/(\eta_1^2) \end{aligned} \quad (10)$$

where $\eta_1 = S_{ij}S_{ij}$ and $\eta_2 = W_{ij}W_{ij}$. Closure modeling is now required for the scalar invariants $\phi_{ij}S_{ij}$, .. etc. In dealing with the invariants, we are more likely to focus on real turbulence physics rather than be bogged down with the coordinate and reference frame transformation issues encountered with tensors.

Opting for a short tensor representation at the expense of more complete model coefficients leads to a very important benefit. If the number of tensors in the representation is much larger than the shortest basis required in physical space, many of the basis tensors will be linearly dependent. This can lead to serious difficulties since their coefficients will also be linearly dependent. The coefficients are typically determined by solving a set of constraint equations. When the number of basis tensors exceeds the minimum value, this coefficient equations become underdetermined and no unique choice is possible. Unless extreme caution is exercised, one can possibly end up with a model which involves cancellation of large terms. Such models will clearly be unsatisfactory.

Alternate representation for the standard model

To demonstrate the viability of the *optimal basis* approach, we will recast the standard four-tensor basis model in a three-tensor basis. Consider again the representation showed earlier:

$$\phi_{ij} = H_1 M^1_{ij} + H_2 M^2_{ij} + H_3 M^3_{ij} \tag{11}$$

where

$$\begin{aligned} M^1_{ij} &= S_{ij} \\ M^2_{ij} &= (S_{ik}W_{kj} - W_{ik}S_{kj}) \\ M^3_{ij} &= (S_{ik}S_{kj} - \frac{1}{3}\delta_{ij}S_{lm}S_{lm}) \end{aligned} \tag{12}$$

Clearly, this representation will be incomplete when the strain-rate tenor or the rotation rate vanish. In all other two-dimensional mean flows, the standard pressure-strain model can be exactly recast in this form. An attractive feature of this representation is that the basis tensors are orthogonal in an invariant sense:

$$M^1_{ij}M^2_{ij} = 0;\ M^1_{ij}M^3_{ij} = 0;\ M^2_{ij}M^3_{ij} = 0. \tag{13}$$

From this it can be easily inferred that

$$\begin{aligned} H_1 &= M^1_{ij}\phi_{ij}/(M^1_{pq}M^1_{pq}) \\ H_2 &= M^2_{ij}\phi_{ij}/(M^2_{pq}M^2_{pq}) \\ H_3 &= M^3_{ij}\phi_{ij}/(M^3_{pq}M^3_{pq}) \end{aligned} \tag{14}$$

We can obtain the three basis tensor equivalent of any pressure-strain correlation model (in two-dimensional mean flow) by merely substituting that model in the above expressions and calculating the corresponding H's.

In deriving the equivalent for the standard model given in equation (1), the following identities are used:

$$\begin{aligned} S_{ik}S_{kj} &= 0.5\eta_1\delta^{(2)}_{ij} \\ W_{ik}W_{kj} &= -0.5\eta_2\delta^{(2)}_{ij} \\ S_{ik}W_{kl}S_{lj} &= -0.5\eta_1 W_{ij} \\ W_{ik}S_{kl}W_{lj} &= 0.5\eta_2 S_{ij} \\ M^2_{ij}M^2_{ij} &= 2\eta_1\eta_2 \\ M^3_{ij}M^3_{ij} &= \eta_1^2/6 \\ I_1 &\equiv b_{ik}S_{kj}S_{ji} \\ I_2 &\equiv b_{ik}S_{kj}W_{ji} \end{aligned} \tag{15}$$

where $\delta^{(2)}_{ij}$ is the two-dimensional Kronecker delta function. The coefficients of the three-tensor version of the standard model is:

$$\begin{aligned} H_1 &= C_2 + C_1\frac{P^2}{\eta_1 \varepsilon K} + 2C_3\frac{I_1}{\eta_1} + 2C_4\frac{I_2}{\eta_2} \\ H_2 &= 2C_2\frac{PI_2}{\eta_1\eta_2\varepsilon} - 2C_4\frac{P}{2K\eta_1} \\ H_3 &= 6C_1\frac{PI_1}{\eta_1^2\varepsilon} + \frac{2}{3}C_3\frac{P}{2K\eta_1} \end{aligned} \tag{16}$$

This alternate representation brings up an interesting question. Rapid distortion theory indicates that the value of the coefficient of S_{ij} must be 0.8. It is for that reason that $C_2 = 0.8$ in the LRR model. In the new representation, the coefficient of S_{ij} is H_1. What is the implication of the Rapid Distortion Theory for H_1? It can be argued that it is H_1 which should be 0.8 and not C_2. This example clearly highlights the perils of including too many linearly dependent basis tensors in the model representation.

Another three-tensor representation with a wider range of applicability is

$$\begin{aligned} M^1_{ij} &= b_{ij} \\ M^2_{ij} &= (b_{ik}W_{kj} - W_{ik}b_{kj}) \\ M^3_{ij} &= (b_{ik}b_{kj} - \frac{1}{3}\delta_{ij}b_{lm}b_{lm}) \end{aligned} \tag{17}$$

DISCUSSION AND CONCLUSION

In this paper we address the issue of how many basis tensors are required for an adequate representation of pressure-strain correlation. The integrity basis, stemming from Representation Theory, sets the upper bound for the number of basis tensors required. The integrity basis completely spans the function space and is guaranteed to be well-behaved at all times. However, even for a small number of candidate tensors (b_{ij}, S_{ij}, W_{ij}) the number of basis tensors exceeds manageable proportions. The dimensionality of the physical space sets the lower limit on the number of basis tensors required. That number is three for two-dimensional mean flows and five in the three-dimensional case.

We also demonstrate in this paper how longer versions of pressure-strain correlation models can be easily recast in terms of smaller tensor basis representation. Clearly the implication is that we really do not need very many of the integrity basis tensors for adequately representing pressure-strain correlation. The shorter representation has another important advantage. The coefficients can be determined with more precision than in the case of longer representation. Based on these, we propose the use of optimal representation which, in number of basis tensors, is closer to the dimensionality limit.

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