THEORETICAL STUDY FOR THE HIGH-ORDER NONLINEAR EDDY-VISCOSITY REPRESENTATION

Masayoshi Okamoto
Department of Mechanical Engineering, Shizuoka University
Hamamatsu, 432-8561, Japan
tmmokam@ipc.shizuoka.ac.jp

Nobuyuki Shima
Department of Mechanical Engineering, Shizuoka University
Hamamatsu, 432-8561, Japan
tmnshim@ipc.shizuoka.ac.jp

ABSTRACT

A general form of an eddy-viscosity representation for the Reynolds stress is a finite tensor polynomial with the fifth power of the mean velocity gradient. In this paper the fifth-order eddy-viscosity representation is investigated with the aid of a two-scale turbulence theory. We apply the nonlinear model to the homogeneous shear flows.

INTRODUCTION

The Reynolds-averaged turbulence model with the nonlinear eddy viscosity has been studied very actively. The second-order nonlinear models were proposed by Speziale (1987), Yoshizawa (1984), Rubinstein and Barton (1990), among others. These models reproduce the anisotropic effect of the Reynolds stress and predict the secondary flow in a square duct flow. Moreover, in recent years, several third-order nonlinear models were proposed by Craft et al. (1996), Shih et al. (1997) and the authors (2000) and the model expressions include the rotation and curvature effects. Pope (1975) studied a nonlinear eddy-viscosity formulation and showed that a general form is a finite tensor polynomial by ten symmetric base tensors composed of mean strain and vorticity tensors. The highest-order nonlinear representation is a fifth-order one. However he did not determine the model coefficients in the general model.

The statistical theories for the turbulence model are a renormalization group (RNG) theory, a two-scale direct-interaction approximation (TSDIA) and so on. The RNG for turbulence was proposed by Yakhot and Orszag (1986). Rubinstein and Barton (1990, 1991) derived the eddy viscosity model using the RNG. The TSDIA was proposed by Yoshizawa (1984, 1987) and several turbulence models with anisotropy, helicity and non-equilibrium effects were suggested using the TSDIA. These theories are complicated two-point closure methods and we can determine model coefficients by the spectral information. However, it is difficult to perform the high-order analysis of these methods. Yoshizawa (1993) suggested a bridging method between the eddy-viscosity-type and stress-transport-type models through the two-scale procedure. The bridging method is a one-point closure one and is a simpler procedure than the TSDIA.

In the present work, the fifth-order representation of the Reynolds stress is theoretically investigated using the bridging method with the previous result of the TSDIA analysis performed by Okamoto (1994). In the third-order eddy-viscosity expression, we compare the present result with the TSDIA one. We test the nonlinear model in three homogeneous shear flows.

MATHEMATICAL PROCEDURES

Fundamental Equations

The Navier-Stokes equation with the incompressible condition is

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (1)$$

$$\frac{\partial u_j}{\partial x_j} = 0, \quad (2)$$

where u_i, p and ν are the velocity, kinematic pressure and kinematic viscosity, respectively. We take the ensemble average of eqs.(1) and (2), and have mean field equations

$$\frac{\partial U_i}{\partial t}+\frac{\partial U_i U_j}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 U_i}{\partial x_j \partial x_j}+\frac{\partial R_{ij}}{\partial x_j}, \quad (3)$$

$$\frac{\partial U_j}{\partial x_j}=0, \quad (4)$$

where U_i and P are the mean velocity and pressure, R_{ij} is the Reynolds stress defined by $R_{ij} \equiv -\overline{u'_i u'_j}$ and u'_i is the fluctuating velocity. The governing equations for u'_i are written by

$$\frac{\partial u'_i}{\partial t}+\frac{\partial u'_i U_j}{\partial x_j}=-\frac{\partial U_i u'_j}{\partial x_j}-\frac{\partial u'_i u'_j}{\partial x_j}-\frac{\partial R_{ij}}{\partial x_j} -\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u'_i}{\partial x_j \partial x_j}, \quad (5)$$

$$\frac{\partial u'_j}{\partial x_j}=0. \quad (6)$$

Two-Scale Formalism

We use a small parameter δ and introduce the space and time variables as

$$\mathbf{x}\,(\equiv \mathbf{x})\,,\ \mathbf{X}\,(\equiv \delta\mathbf{x})\,,\ t\,(\equiv t)\,,\ T\,(\equiv t)\,. \quad (7)$$

The mean field is dependent on the slow variables and the fluctuating one depends on the slow and fast variables. The mean and fluctuating quantities are expressed by

$$f=F(\mathbf{X},T)+f'(\mathbf{x},\mathbf{X},t,T) \quad (8)$$

We apply this scale separation (8) to the fluctuating field equations (5) and (6) and solve the fluctuating equations perturbatively by a direct-interaction approximation proposed by Kraichnan (1964).

As a result, the second-order expression of the Reynolds stress R_{ij} in the TSDIA analysis is written as

$$R_{ij}+\frac{2}{3}K\delta_{ij}=\gamma_1 S_{ij}+\gamma_2\frac{DS_{ij}}{Dt} +\gamma_3\left(S_{im}S_{mj}\right)^* +\gamma_4\left(S_{im}W_{mj}+S_{jm}W_{mi}\right), \quad (9)$$

with

$$\gamma_1=0.123\frac{K^2}{\varepsilon}-0.147\frac{K^2}{\varepsilon^2}\frac{DK}{Dt} +0.0933\frac{K^3}{\varepsilon^3}\frac{D\varepsilon}{Dt}, \quad (10)$$

$$\gamma_2=-0.0427\frac{K^3}{\varepsilon^2}, \quad (11)$$

$$\gamma_3=-0.0297\frac{K^3}{\varepsilon^2}, \quad (12)$$

$$\gamma_4=0.0122\frac{K^3}{\varepsilon^2}, \quad (13)$$

where K and ε are the turbulence energy and its dissipation rate, $S_{ij} \equiv \partial U_i/\partial x_j+\partial U_j/\partial x_i$, $W_{ij} \equiv \partial U_i/\partial x_j-\partial U_j/\partial x_i$, D/Dt is the Lagrangian derivative and $*$ indicates the deviatoric part of a tensor defined by $(S_{im}S_{mj})^*=S_{im}S_{mj}-S_{mn}S_{nm}\delta_{ij}/3$. The model constant in the linear eddy-viscosity term is larger than 0.09 in the standard K-ε model and the coefficient includes non-equilibrium factors DK/Dt and $D\varepsilon/Dt$ pointed out by Yoshizawa and Nisizima (1993). The details of this TSDIA analysis are described by Okamoto (1994).

Bridging Method

In this section, we give an outline of the bridging method between the eddy viscosity and stress models through the two-scale procedure. By renormalizing the second-order expression (9), a governing equation leading to the asymptotic solution (9) is obtained as

$$\begin{aligned} B_{ij}=&\,0.123\frac{K^2}{\varepsilon}S_{ij}-0.347\frac{K}{\varepsilon}\frac{DB_{ij}}{Dt}\\ &-0.121\frac{K}{\varepsilon}\left(B_{im}S_{mj}+S_{im}B_{mj}\right)^*\\ &+0.100\frac{K}{\varepsilon}\left(B_{im}W_{mj}+B_{jm}W_{mi}\right)\\ &-0.0616\frac{K^2}{\varepsilon^2}\frac{DK}{Dt}S_{ij}\\ &+0.0506\frac{K^3}{\varepsilon^3}\frac{D\varepsilon}{Dt}S_{ij} \end{aligned} \quad (14)$$

where B_{ij} is the deviatoric part of the Reynolds stress defined by $B_{ij} \equiv R_{ij}+2K\delta_{ij}/3$. We solve eq. (14) in perturbational manner and obtain the higher-order eddy viscosity model for the Reynolds stress.

At first, let us check difference between results of the TSDIA and the present analysis with respect to the third-order expression of R_{ij}. The third-order nonlinear form is expressed by

$$\begin{aligned} B_{ij}=&\,\gamma_1 T^{(1)}_{ij}+\gamma_2\dot{T}^{(1)}_{ij}+\gamma_3 T^{(2)*}_{ij}+\gamma_4 T^{(3)}_{ij}\\ &+\gamma_5 T^{(4)*}_{ij}+\gamma_6\ddot{T}^{(1)}_{ij}+\gamma_7\dot{T}^{(2)*}_{ij}+\gamma_8\dot{T}^{(3)}_{ij}\\ &+\gamma_9\dot{T}^{(4)*}_{ij}+\gamma_{10}\left(\dot{S}_{im}W_{mj}+\dot{S}_{jm}W_{mi}\right)\\ &+\gamma_{11}C^{(1)}T^{(1)}_{ij}+\gamma_{12}C^{(2)}T^{(1)}_{ij}+\gamma_{13}T^{(6)}_{ij}\\ &+\gamma_{14}T^{(7)*}_{ij}. \end{aligned} \quad (15)$$

Table 1: Model constants of the TSDIA and the present analysis. The bold-faced values are the TSDIA ones.

	TSDIA	Present analysis
Main coefficient		
γ_1	**0.123**	**0.123**
γ_2	**-0.0427**	**-0.0427**
γ_3	**-0.0297**	**-0.0297**
γ_4	**0.0122**	**0.0122**
γ_5	**0**	**0**
γ_6	**0.0169**	0.0148
γ_7	**0.0176**	0.0155
γ_8	**0.00487**	0.00423
γ_9	**0**	0
γ_{10}	**-0.00480**	-0.00424
γ_{11}	**-0.00307**	0.00360
γ_{12}	**0.00803**	0.00253
γ_{13}	**-0.00350**	-0.00443
γ_{14}	**0.00523**	0.00363
DK/Dt coefficient		
γ_1	**-0.147**	**-0.147**
γ_2	**0.107**	0.0955
γ_3	**0.0754**	0.0666
γ_4	**-0.0316**	-0.0273
γ_5	**0**	0
$D\varepsilon/Dt$ coefficient		
γ_1	**0.0933**	**0.0933**
γ_2	**-0.0640**	-0.0620
γ_3	**-0.0460**	-0.0432
γ_4	**0.0194**	0.0177
γ_5	**0**	0

Here, $\dot{T} \equiv DT/Dt$ and the tensors $T_{ij}^{(n)}$ are symmetric base tensors given in Appendix. The model coefficients are

$$\gamma_1 = C_{main}\frac{K^2}{\varepsilon} + C_{DK/Dt}\frac{K^2}{\varepsilon^2}\frac{DK}{Dt} + C_{D\varepsilon/Dt}\frac{K^3}{\varepsilon^3}\frac{D\varepsilon}{Dt}, \quad (16)$$

$$\gamma_{2\sim5} = C_{main}\frac{K^3}{\varepsilon^2} + C_{DK/Dt}\frac{K^3}{\varepsilon^3}\frac{DK}{Dt} + C_{D\varepsilon/Dt}\frac{K^4}{\varepsilon^4}\frac{D\varepsilon}{Dt}, \quad (17)$$

$$\gamma_{6\sim14} = C_{main}\frac{K^4}{\varepsilon^3}. \quad (18)$$

The model constants in γ_n are summarized in Table 1. The values of this analysis are in agreement with those of the TSDIA, though there are disagreements in γ_{11} and γ_{12} between both the results. The terms of γ_{11} and γ_{12} are proportional to the linear eddy-viscosity term and we can incorporate the terms into γ_1. Therefore, this comparison result shows that the model constants of the main coefficients by the bridging method correspond with those by the TSDIA.

Next, we show the fifth-order eddy-viscosity representation of the Reynolds stress. For the purpose of obtaining a simple expression, we apply the Cayleigh-Hamilton theorem to the result of the present analysis and neglect the terms with the Lagrangian derivative. We have the fifth-order nonlinear expression

$$B_{ij} = \Gamma_1 T_{ij}^{(1)} + \Gamma_2 T_{ij}^{(2)*} + \Gamma_3 T_{ij}^{(3)} + \Gamma_4 T_{ij}^{(4)*} + \Gamma_5 T_{ij}^{(5)} + \Gamma_6 T_{ij}^{(6)*} + \Gamma_7 T_{ij}^{(7)} + \Gamma_8 T_{ij}^{(8)*} + \Gamma_9 T_{ij}^{(9)} + \Gamma_{10} T_{ij}^{(10)}, \quad (19)$$

with

$$\Gamma_1 = C_1\frac{K^2}{\varepsilon}f_1, \quad (20)$$

$$\Gamma_{2,3} = C_{2,3}\frac{K^3}{\varepsilon^2}f_{2,3}, \quad (21)$$

$$\Gamma_4 = C_4\frac{K^3}{\varepsilon^2} + f_4, \quad (22)$$

$$\Gamma_{5,6} = C_{5,6}\frac{K^4}{\varepsilon^3}f_{5,6}, \quad (23)$$

$$\Gamma_{7\sim9} = C_{7\sim9}\frac{K^5}{\varepsilon^4}, \quad (24)$$

$$\Gamma_{10} = C_{10}\frac{K^6}{\varepsilon^5}. \quad (25)$$

This tensor polynomial (19) was found by Pope (1975), but the entire set of coefficients was not determined by anyone. The result of main constants C_n is shown in Table 2. In this result, the main constants of Γ_4 and Γ_{10} are zero. The former indicates that this expression satisfies the frame invariance pointed out by Speziale (1979, 1981). The model functions in eqs.(20) - (23) are written as

$$f_1 = 1 + \frac{1}{6}C_S^2\tau^2 C^{(1)} - C_W^2\tau^2 C^{(2)} + C_S C_W^2\tau^3 C^{(4)} + \frac{1}{36}C_S^4\tau^4 C^{(1)2} + \frac{19}{24}C_S^2 C_W^2\tau^4 C^{(1)}C^{(2)} - 2C_S^2 C_W^2\tau^4 C^{(5)} - \frac{7}{2}C_W^4\tau^4 C^{(2)2}, \quad (26)$$

$$f_2 = 1 + \frac{1}{6}C_S^2\tau^2 C^{(1)} + \frac{1}{2}C_W^2\tau^2 C^{(2)} + 4C_S C_W^2\tau^3 C^{(4)}, \quad (27)$$

$$f_3 = 1 + \frac{13}{24}C_S^2\tau^2 C^{(1)} + \frac{1}{2}C_W^2\tau^2 C^{(2)} - \frac{1}{8}C_S^3\tau^3 C^{(3)} + \frac{5}{2}C_S C_W^2\tau^3 C^{(4)}, \quad (28)$$

$$f_4 = -\frac{3}{2}C_S C_W^2\gamma_1'\tau^3 C^{(1)} + \frac{1}{2}C_S^2 C_W^2\gamma_1'\tau^4 C^{(3)} + 6C_W^4\gamma_1'\tau^4 C^{(4)}, \quad (29)$$

Table 2: Model constants of main coefficients.

C_1	C_2	C_3	C_4
0.123	-0.0298	0.0123	0
C_5	C_6	C_7	C_8
-0.00446	0.00369	0.000540	-0.000893
C_9	C_{10}		
-0.000369	0		

$$f_5 = 1 + \frac{7}{24} C_S^2 \tau^2 C^{(1)} + \frac{5}{8} C_W^2 \tau^2 C^{(2)}, \quad (30)$$

$$f_6 = 1 + \frac{7}{24} C_S^2 \tau^2 C^{(1)} + \frac{5}{2} C_W^2 \tau^2 C^{(2)}, \quad (31)$$

where τ is a turbulence time-scale K/ε, $C_S = -0.242$, $C_W = 0.100$, $\gamma_1' = 0.123K^2/\varepsilon$ and $C^{(n)}$ are the invariants given in Appendix. The functions f_n arise from the high-order expansion terms and the functional forms are complicated. We cannot find any simple rule among f_n of the present result, and the model functions in the same expansion order are different. In the present analysis, many terms with the Lagrangian derivative are derived. If we introduce the terms in the eddy-viscosity representation, the fifth-order expression is not a closed set of R_{ij}.

APPLICATION TO HOMOGENEOUS SHEAR FLOWS

Finally, we apply the present expression with a standard set of K and ε equations to three homogeneous shear flows. The model functions used in the present model are

$$f_1 = \left(1 + 0.006\tau^2 S_{ab} S_{ab}\right)^{-1}, \quad (32)$$

$$f_6 = \left(1 + 0.022\tau^2 S_{ab} S_{ab}\right)^{-2}, \quad (33)$$

tentatively. The initial nondimensional shear rates S_0 are 3.38 in the large eddy simulation of Bardina et al. (1983), 6.47 in the experiment of Tavoularis and Corrsin (1981) and 50.0 in the direct numerical simulation of Lee et al. (1990). Figure 1 shows the numerical results. The standard K-ε model overpredicts the turbulence energy in all the cases. Speziale (1996) showed that in the case of $S_0 = 50.0$ several stress models overpredict the turbulence energy like the standard K-ε model and pointed out that the case is a strong nonequilibrium case. The present model is in good agreement with the reference data in all the cases.

CONCLUSION

In this work, we derived the fifth-order eddy-viscosity representation perturbatively by the bridging theory with the second-order results of the TSDIA analysis. The result gives us some information related to the nonlinear eddy-viscosity model.

APPENDIX: TENSOR ANALYSIS

In three dimensions the symmetric base tensors constituted by S_{ij} and W_{ij} are

$$T^{(1)}_{\alpha\beta} = S_{\alpha\beta}, \quad (34)$$

$$T^{(2)}_{\alpha\beta} = S_{\alpha a} S_{a\beta}, \quad (35)$$

$$T^{(3)}_{\alpha\beta} = S_{\alpha a} W_{a\beta} + S_{\beta a} W_{a\alpha}, \quad (36)$$

$$T^{(4)}_{\alpha\beta} = W_{\alpha a} W_{a\beta}, \quad (37)$$

$$T^{(5)}_{\alpha\beta} = S_{\alpha a} S_{ab} W_{b\beta} + S_{\beta a} S_{ab} W_{b\alpha}, \quad (38)$$

$$T^{(6)}_{\alpha\beta} = S_{\alpha a} W_{ab} W_{b\beta} + S_{\beta a} W_{ab} W_{b\alpha}, \quad (39)$$

$$T^{(7)}_{\alpha\beta} = S_{\alpha a} S_{ab} W_{bc} S_{c\beta} + S_{\beta a} S_{ab} W_{bc} S_{c\alpha}, \quad (40)$$

$$T^{(8)}_{\alpha\beta} = S_{\alpha a} S_{ab} W_{bc} W_{c\beta} + S_{\beta a} S_{ab} W_{bc} W_{c\alpha}, \quad (41)$$

$$T^{(9)}_{\alpha\beta} = W_{\alpha a} S_{ab} W_{bc} W_{c\beta} + W_{\beta a} S_{ab} W_{bc} W_{c\alpha}, \quad (42)$$

$$T^{(10)}_{\alpha\beta} = W_{\alpha a} S_{ab} S_{bc} W_{cd} W_{d\beta} + W_{\beta a} S_{ab} S_{bc} W_{cd} W_{d\alpha}, \quad (43)$$

and the invariants are

$$C^{(1)} = S_{ab} S_{ba}, \quad (44)$$

$$C^{(2)} = W_{ab} W_{ba}, \quad (45)$$

$$C^{(3)} = S_{ab} S_{bc} S_{ca}, \quad (46)$$

$$C^{(4)} = S_{ab} W_{bc} W_{ca}, \quad (47)$$

$$C^{(5)} = S_{ab} S_{ba} W_{cd} W_{dc}. \quad (48)$$

The Cayleigh-Hamilton theory in three dimensions is expressed by

$$\begin{aligned}
& A_{\alpha a} B_{ab} C_{b\beta} + B_{\alpha a} C_{ab} A_{b\beta} + C_{\alpha a} A_{ab} B_{b\beta} \\
& + B_{\alpha a} A_{ab} C_{b\beta} + A_{\alpha a} C_{ab} B_{b\beta} + C_{\alpha a} B_{ab} A_{b\beta} \\
& = A_{aa} \left(B_{\alpha b} C_{b\beta} + C_{\alpha b} B_{b\beta} \right) \\
& + B_{aa} \left(C_{\alpha b} A_{b\beta} + A_{\alpha b} C_{b\beta} \right) \\
& + C_{aa} \left(A_{\alpha b} B_{b\beta} + B_{\alpha b} A_{b\beta} \right) \\
& + \left(B_{ab} C_{ba} - B_{aa} C_{bb} \right) A_{\alpha\beta} \\
& + \left(C_{ab} A_{ba} - C_{aa} A_{bb} \right) B_{\alpha\beta} \\
& + \left(A_{ab} B_{ba} - A_{aa} B_{bb} \right) C_{\alpha\beta} \\
& + (A_{aa} B_{bb} C_{cc} - A_{aa} B_{bc} C_{cb} \\
& - B_{aa} C_{bc} A_{cb} - C_{aa} A_{bc} B_{cb} \\
& + A_{ab} B_{bc} C_{ca} + C_{ab} B_{bc} A_{ca}) \delta_{\alpha\beta}. \quad (49)
\end{aligned}$$

Here A_{ij}, B_{ij} and C_{ij} are arbitrary tensors in three dimensions. The details of the tensor analysis are reported by Spencer and Rivlin (1959, 1960).

References

Bardina, J., Ferziger, J.M. and Reynolds, W.C., 1983, "Improved turbulence models based on large-eddy simulation of homogeneous incompressible turbulent flows", Report No.TF-19, Stanford University.

Craft, T.J., Launder, B.E. and Suga, K., 1996, "Development and application of a cubic eddy-viscosity model of turbulence", *Int. J. Heat Fluid Flow*, Vol. 17, pp. 108-115.

Kraichnan, R.H., 1964, "Direct-Interaction Approximation for Shear and Thermally Driven Turbulence", *Phys. Fluids*, Vol. 7, pp. 1048-1062.

Lee, M.J., Kim, J. and Moin, P., 1990, "Structure of turbulence at high shear rate", *J. Fluid Mech.*, Vol. 216, pp. 561-583.

Okamoto, M., 1994, "Theoretical Investigation of an Eddy-Viscosity-Type Representation of the Reynolds Stress", *J. Phys. Soc. Jpn.*, Vol. 63, pp. 2102-2122.

Okamoto, M. and Shima, N., 2000, "A Nonlinear K-ε Model With a Third-Order Eddy-Viscosity Representation", *Proceedings, Turbulence, Heat and Mass Transfer 3*, Nagano, Y. et al., ed., Aichi Shuppan, Tokyo, pp. 389-396.

Pope, S., 1975, "A more general effective-viscosity hypothesis", *J. Fluid Mech.*, Vol. 72, pp. 331-340.

Rubinstein, R. and Barton, J.M., 1990, "Nonlinear Reynolds stress models and the renormalization group", *Phys. Fluids A*, Vol. 2, pp. 1472-1476.

Rubinstein, R. and Barton, J.M., 1991, "Renormalization group analysis of anisotropic diffusion in turbulent shear flows", *Phys. Fluids A*, Vol. 3, pp. 415-421.

Shih, T.H., Zhu, J. and Lumley, J.L., 1993, "A Realizable Reynolds Stress Algebraic Equation Model", Technical Report TM 105993, NASA.

Spencer, A.J.M. and Rivlin, R.S., 1959, "The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua", *Arch. Rat. Mech. Anal.*, Vol. 2, pp. 309-336.

Spencer, A.J.M. and Rivlin, R.S., 1960, "Further Results in the Theory of Matrix Polynomials", *Arch. Rat. Mech. Anal.*, Vol. 4, pp. 214-230.

Speziale, C.G., 1979, "Invariance of Turbulent closure models", *Phys. Fluids*, Vol. 178, pp. 459-475.

Speziale, C.G., 1981, "Some interesting properties of two-dimensional turbulence", *Phys. Fluids*, Vol. 22, pp. 1033-1037.

Speziale, C.G., 1987, "On nonlinear $k-l$ and $k-\varepsilon$ models of turbulence", *J. Fluid Mech.*, Vol. 24, pp. 1425-1427.

Speziale, C.G., 1996, "Modeling of turbulent transport equations", *Simulation and Modeling of Turbulent Flows*, Gatski, T.B. et al., ed., Oxford University Press, Inc., New York, pp. 185-242.

Tavouralis, S. and Corrsin, S., 1981, "Experiments in nearly homogeneous turbulent shear flow with a uniform mean temperature gradient. Part 1", *J. Fluid Mech.*, Vol. 104, pp. 311-347.

Yoshizawa, A., 1984, "Statistical analysis of the deviation of the Reynolds stress from its eddy viscosity representation", *Phys. Fluids*, Vol. 27, pp. 1377-1387.

Yoshizawa, A., 1987, "Statistical modeling of a transport equation for the kinetic energy dissipation", *Phys. Fluids*, Vol. 30, pp. 628-631.

Yoshizawa, A., 1993, "Bridging between eddy-viscosity-type and second-order turbulence models through a two-scale turbulence theory", *Phys. Rev. E*, Vol. 48, pp. 273-281.

Yoshizawa, A. and Nisizima S., 1993, "A nonequilibrium representation of the turbulent viscosity based on a two-scale turbulence theory", *Phys. Fluids A*, Vol. 5, pp. 3302-3304.

Yakhot, V. and Orszag, S.A., 1986, "Renormalization group analysis of turbulence I. Basic theory", *J. Sci. Comp.*, Vol. 1, pp. 3-51.

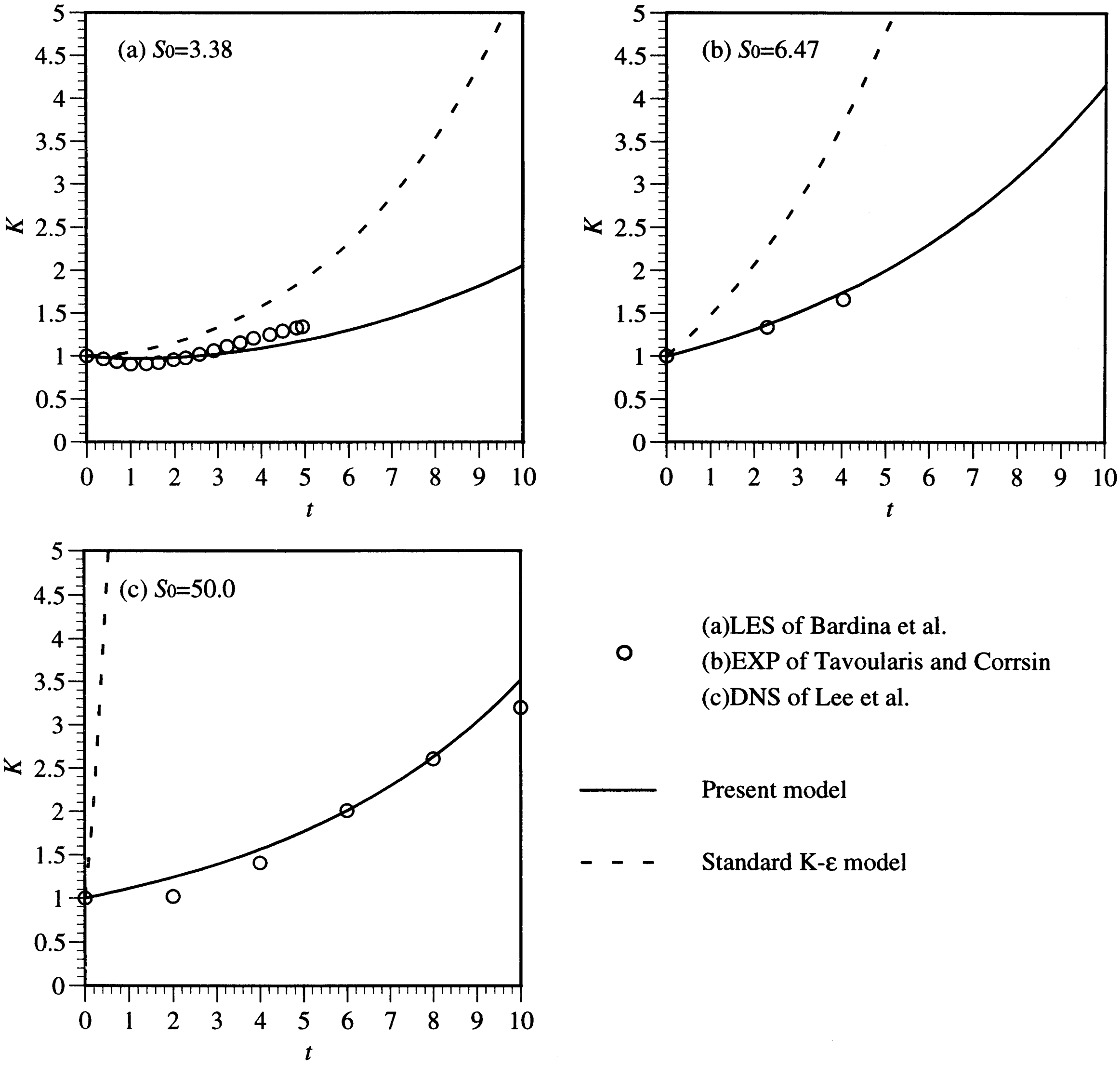

Figure 1: Turbulence energy in homogeneous shear flows.