NUMERICAL SIMULATIONS OF THE LAGRANGIAN AVERAGED NAVIER-STOKES (LANS-α) EQUATIONS

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ABSTRACT

We examine the modeling capabilities of the Lagrangian-Averaged Navier-Stokes (LANS-α) equations for decaying homogeneous turbulence. Direct Numerical Simulation (DNS) of the Navier-Stokes equations is performed with an initial Taylor Reynolds number of 220. The predictive capability of the LANS-α equations is analyzed by comparison of the numerical solution of the LANS-α equations with the Large Eddy Simulation (LES) and the DNS data. The LANS-α equations accurately predict the total kinetic energy decay and the time evolution of the energy spectra as long as a minimum resolution is met.

INTRODUCTION

Isotropic turbulence has attracted much interests in the turbulence community mainly because there is evidence that high Reynolds number turbulent flows can be locally isotropic (Seyed and Veeravalli 1994). Many features of a turbulent flow, such as viscous dissipation and vortex stretching are present in homogeneous turbulence. Direct Numerical Simulation (DNS) of homogeneous isotropic turbulence is well established and has been extensively studied in the last three decades. Such computations can be used to probe features, examine statistics of quantities normally not available in conventional experiments, and test new turbulence models. The main difficulty in the turbulence engineering community is that performing the DNS of typical engineering problems (usually at high Reynolds numbers) is very expensive, and therefore unlikely to happen in the foreseeable future. This is primarily due to the fact that the number of degrees of freedom for three-dimensional Navier-Stokes flow grows rapidly with Reynolds number; it is proportional to $\mathrm{Re}^{9/4}$.

There are methods for simulating turbulent flows where one does not need to use the brute-force approach in the DNS to resolve all scales of motion. A popular alternative is the Large Eddy Simulation (LES) in which only the large scales of motion are resolved while the effects of small scales on the large scales are modeled (see *e.g.*, Moin 1998). The basic idea behind LES is to define a large scale field through a low-pass filtering of the flow variables; subsequently, the governing equations for the mean flow quantities (large scales) are obtained by filtering the Navier-Stokes and continuity equations.

Another method is the Reynolds Averaging of the Navier-Stokes equations (RANS). In this approach the flow field is decomposed into a time- or ensemble-averaged mean flow and a fluctuating perturbation field. Substitution of this field decomposition into the Navier-Stokes equations results in a set of differential equations for the mean flow quantities containing contributions from the time-varying, turbulent motion. This requires the introduction of a turbulence model to describe the effect of these fluctuations on the mean quantities.

In this study, we consider a new approach introduced in Holm *et al.* (1998) and Marsden and Shkoller (2001). Unlike the traditional averaging or filtering approach used for both the RANS and the LES, wherein the Navier-

Stokes equations are averaged, the LANS-α approach is based on averaging at the level of the variational principle from which the Euler equations are derived. Namely, a new averaged action principle is defined. The least action principle then yields the so-called Lagrangian-Averaged Euler (LAE-α) equations. The Lagrangian-Averaged Navier-Stokes equations (LANS-α) are obtained by the addition of the usual viscous terms to the LAE-α equations. An anisotropic version of the LANS-α equations was recently developed by Marsden and Shkoller (2001).

In our previous study (Mohseni *et al.* 2000) we performed the DNS, LES, and LANS-α computations for simulating a case comparable with the decaying grid turbulence experiment by Comte-Bellot and Corrsin (1966) and (1971) which we will hereafter refer to as CBC. The LES and LANS-α computations usually perform well when the energy-containing part of the velocity field is well captured. The initial field for CBC simulation was taken from Wray (1998). Wray provided a filtered velocity field in physical space, derived from 512^3 data by a sharp truncation in the Fourier space to 128^3. Therefore the resolution of the DNS computations in Mohseni *et al.* (2000) was limited to 128^3. This initial field was relatively broad with the peak of the energy spectrum at a relatively high wave number (see figure 2 in Mohseni *et al.* 2000). The best behavior of the LANS-α modeling (or many other turbulent models) is expected for applications in which the scale α is within, or at least close to, an inertial subrange. The CBC experiments at $\mathrm{Re}_\lambda = 72$ in a computational box with resolution 128^3 barely satisfy this criterion. To avoid such limitations, in this study we perform a new DNS computation of a decaying isotropic homogeneous turbulence with an energy spectrum peaked at the wavenumber 3, and the resolution of the DNS computation is increased to 170^3 (258^3 before dealiasing). In this manuscript we present results for the decaying turbulence. Results on two sets of forced homogeneous turbulence will be reported in Mohseni *et al.* (2001).

LAGRANGIAN-AVERAGED NAVIER-STOKES EQUATIONS

A detailed derivation of the isotropic and anisotropic LAE-α/LANS-α equations can be found in the article by Marsden and Shkoller (2001). However, a brief summary of the relevant background material on the Lagrangian averaging approach and the resulting Lagrangian averaged Euler (LAE-α) and the Lagrangian-averaged Navier-Stokes (LANS-α) equations are given in Mohseni *et al.* (2001).

The isotropic LANS-α equations in a periodic box are given by

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathrm{Div}\,\mathcal{S}^\alpha(\mathbf{u}) = -\,\mathrm{grad}\,p + \nu\Delta\mathbf{u},$$
$$\mathrm{div}\,\mathbf{u} = 0, \qquad (1)$$

where $\mathbf{u}$ is the macroscopic velocity and ν is the kinematic viscosity. $\mathcal{S}^\alpha(\mathbf{u})$ is the *subgrid stress tensor* defined by

$$\mathcal{S}^\alpha(\mathbf{u}) = \alpha^2(1-\alpha^2\Delta)^{-1}\left[\nabla\mathbf{u}\cdot\nabla\mathbf{u}^T + \nabla\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\mathbf{u}^T\cdot\nabla\mathbf{u}\right]. \qquad (2)$$

In this equation, α is a length scale of the rapid fluctuations in the flow map, below which wave activity is filtered by the nonlinear dispersion. As with the usual Euler equations, the function p is determined from the incompressibility condition, the pressure satisfies a Poisson equation that is determined by taking the divergence of the momentum equation.

Our Lagrangian averaging methodology permits us to use a Lagrangian generalization of G.I. Taylor's *frozen fluid hypothesis* Taylor (1938) as a turbulence closure upto $O(\alpha^3)$ in our averaging parameter α. This is made possible by averaging and asymptotically expanding at the level of the variational principle, instead of at the level of the equations as in the LES and RANS approaches, and considering the Lagrangian fluctuations to be frozen into the mean flow on those time scales which are required to compute temporal rates of change (time derivatives). The Taylor hypothesis is commonly invoked to compute spatial gradients in fluid turbulence experiments, and provides a natural turbulent closure in our Lagrangian averaging framework.

NUMERICAL METHOD

In this study we focus on the numerical solution of decaying homogeneous turbulence. Our computational domain is a periodic cubic box of side 2π. The spatial resolution $k_{max}\eta$ is kept to be greater than 1 to ensure good spatial resolution, where k_{max} is the maximum wavenumber in the simulation (the smallest resolved scale in the simulation) and η is the Kolmogorov length scale representing the dissipation scale.

The full range of scales in a turbulent flow for even a modest Reynolds number spans

many orders of magnitude, and it is not generally feasible to capture them all in a numerical simulation. On the other hand, in turbulence modeling, empirical or theoretical models are used to account for the net effect of small scales on large energy-containing scales. In the next section, the numerical simulations of decaying homogeneous turbulence based on the full DNS, LES modeling, and the LANS-α modeling are presented.

The core of the numerical method used in this study is based on a standard parallel pseudospectral scheme with periodic boundary conditions similar to the one described in Rogallo (1981). The spatial derivatives are calculated in the Fourier domain, while the nonlinear convective terms are computed in the physical space. The flow fields are advanced in time in physical space using a fourth order Runge-Kutta scheme. The time step was chosen appropriately to ensure numerical stability. To eliminate the aliasing errors in this procedure the two thirds rule is used, so that the upper one third of wave modes is discarded at each stage of the fourth order Runge-Kutta scheme.

The LES computations presented in this study are based on the dynamic subgrid scale model of Germano *et al.* (1994). They suggested a dynamic procedure in which the model coefficient of an arbitrary functional relationship, selected to represent the subgrid scale stress tensor, can be evaluated as part of the simulation. This procedure, applied to the Smagorinsky eddy-viscosity model, has proven quite versatile and is used here as a representative of a class of LES models. The filter aspect ratio in the dynamic model is a free parameter and the final result depends on the value of this parameter. In order to avoid introducing any further arbitrary parameters, no averaging operation is performed on the model coefficients over the computational domain. However, the LES computations were repeated for various filter aspect ratios, and the parameter that matched the best with the turbulence decay of the DNS data was used in the computations of the next section.

RESULTS

In this section, we present results of our numerical simulations of the LANS-α equations. The dispersive characteristic of the energy cascade in the LAE-α equations was demonstrated by Mohseni *et al.* (2000), where they found that the LAE-α equations redistribute the energy content among the small scales through a nonlinear dispersive mechanism. The L_2 norm of the energy function is defined by

$$E_{L_2}(\mathbf{u}) = \frac{1}{2}\int \mathbf{u}\cdot\mathbf{u}\, d^3x. \qquad (3)$$

The absolute drop in $E_{L_2}(\mathbf{u})$ in the LAE-α calculations depends on the initial velocity field as well as the value of α. We remark that when α goes to zero, by definition the LANS-α computations approach DNS.

Lagrangian-Averaged Navier-Stokes Equations

In a viscous computation the dispersive decay in $E_{L_2}(\mathbf{u})$ is accompanied by a viscous dissipation, as the viscous effects remove energy from the small scales. In this section viscous computations are performed to quantify the nature of the viscous decay.

To avoid such limitations, in this study we perform a DNS computation of a decaying isotropic homogeneous turbulence with an energy spectrum peaked at the wavenumber 3, and the resolution of the DNS computation is increased to 170^3 (258^3 before dealiasing). In this manuscript we present results for the decaying turbulence. Results on two sets of forced homogeneous turbulence will be reported in Mohseni *et al.* (2001).

We start with a divergence free velocity field with a specified energy spectrum given by

$$E(k) = Ak^4 \exp^{-2k^2/k_0^2}, \qquad (4)$$

where k is the wave number, k_0 is the wave number of the peak of the energy spectrum, and A is a normalizing factor. In all of the computations presented in this study we use $k_0 = 3$, and we choose A such that the total kinetic energy of the initial velocity field is 0.5. The resolution of the DNS computation is 170^3, or 258^3 before dealiasing. The initial Taylor Reynolds number is $\mathrm{Re}_\lambda = 220$. All of the computations presented here were carried out on the Hewlett-Packard Exemplar V2500 at the California Institute of Technology.

The evolution of the energy spectrum in the DNS using 170^3 points is illustrated in figure 1. The energy of fully developed isotropic turbulence decays in time while the scales of motion grow. As a result of the total kinetic energy (TKE) decay the resulting Re_λ decreases with time. Consequently, a well resolved, fully developed field will remain well resolved as the

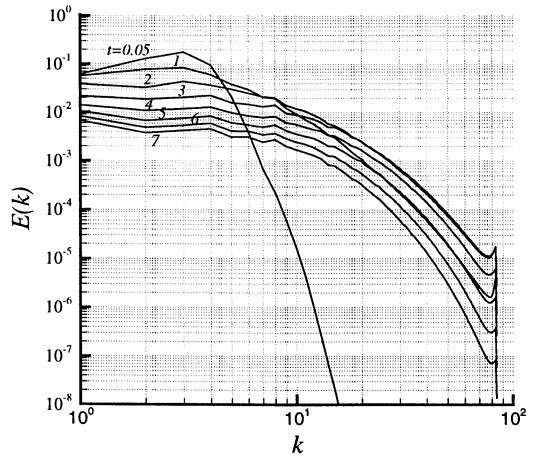
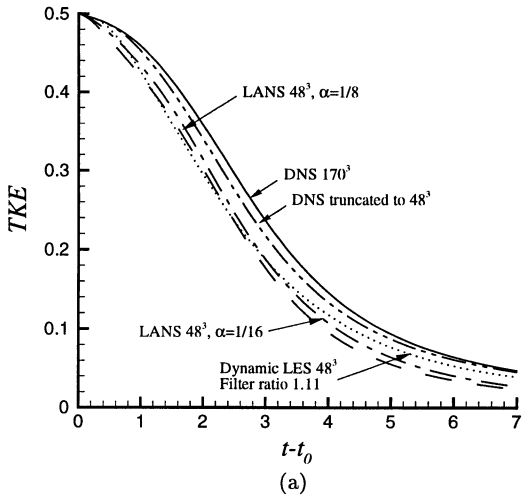


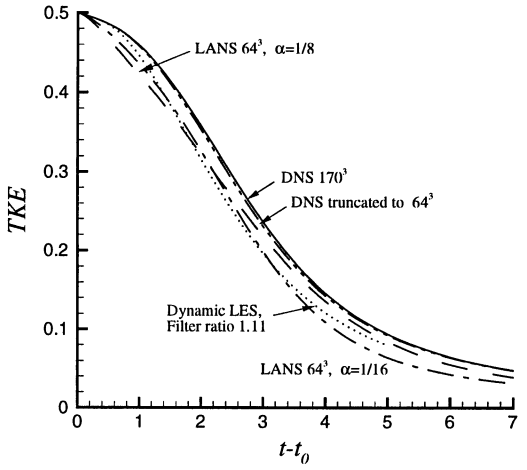
Figure 1: Spectrum of the energy, $\mathrm{Re}_\lambda = 220$, $N^3_{DNS} = 170^3$.

TKE decays. On the other hand, the integral scales grow in time and will eventually become comparable to the size of the computational box. Since the computational box contains only a small sample of the largest representable eddies, eventually the computation will suffer from a lack of sample in the energy-containing scales.

In figure 2 the evolution of the TKE of the DNS data is contrasted against various dynamic LES and the LANS-α simulations, for two resolutions: 48^3 and 64^3. TKE's for the DNS data, sharply filtered to the resolution of the LES and the LANS-α computations, are also presented. Dynamic LES computations for filter aspect ratios of 1.11, 1.33, 2, and 4 are performed. We found that the best match between the DNS data and the dynamic LES results is achieved for a filter aspect ratio of 1.11 for both 48^3 and 64^3 calculations. In both the 48^3 and 64^3 computations the LES and the LANS-α satisfactorily predict the decay rate. This should be contrasted with the results for the simulation of the CBC experiment in Mohseni *et al.* (2000) where a low resolution calculation at 48^3 was significantly in error. This confirms the speculation that for an accurate calculation based on the LANS-α or LES the bulk part of the energy containing wave modes should be included in the simulation. The same conclusion is drawn from figures 3-5 where the evolution of the energy spectrum of various computations are presented. All 48^3 simulations indicate insufficient dissipation of energy at later times due to inadequate resolution. This is more pronounced in the LANS-α computations with



(a)



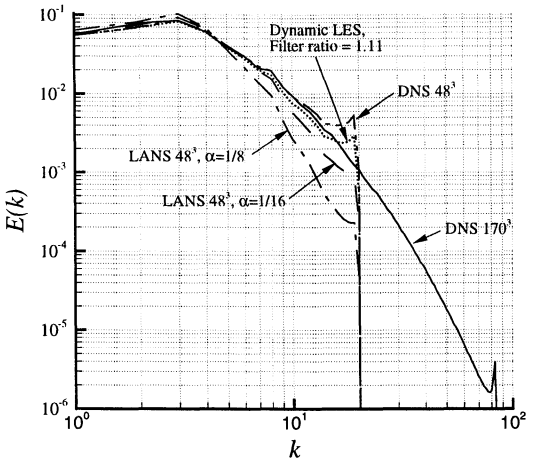
(b)

Figure 2: Decay of kinetic energy; (a) $N^3_\alpha = 48^3$; (b) $N^3_\alpha = 64^3$.

$\alpha = 1/16$ where the model is heavily dependent on the nonlinear dispersive mechanism of the Lagrangian-averaged equations (activated at a higher wavenumber) as opposed to the dissipative model in LES. For 64^3 calculations the energy spectrum is predicted reasonably well by both the LANS-α and the dynamic LES.

CONCLUDING REMARKS

The essence of the Lagrangian averaging method is the dispersive (not dissipative) nature of the energy decay in the Lagrangian-Averaged Euler (LAE-α) equations, where the energy is removed from the small scales while maintaining the crucial features of the large scale flow. In the viscous counterpart (the LANS-α equations) the dispersive decay is accompanied by a viscous dissipation. There-



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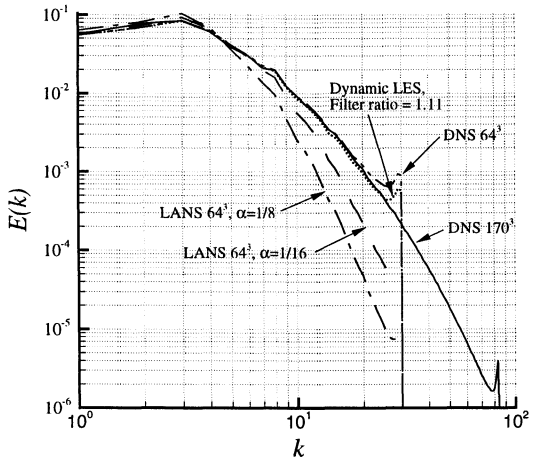
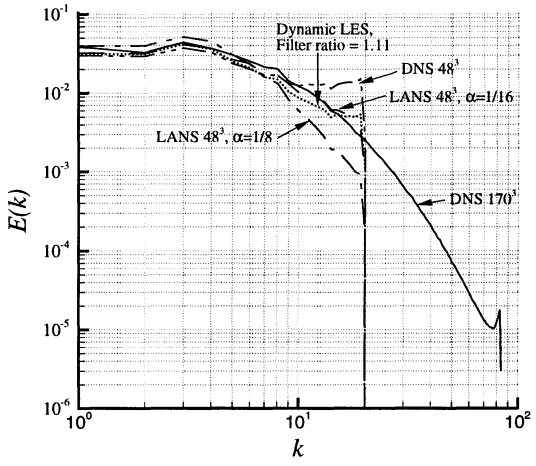


(b)

Figure 3: The energy spectra at $t = 1$; (a) $N_\alpha^3 = 48^3$ (b) $N_\alpha^3 = 64^3$.



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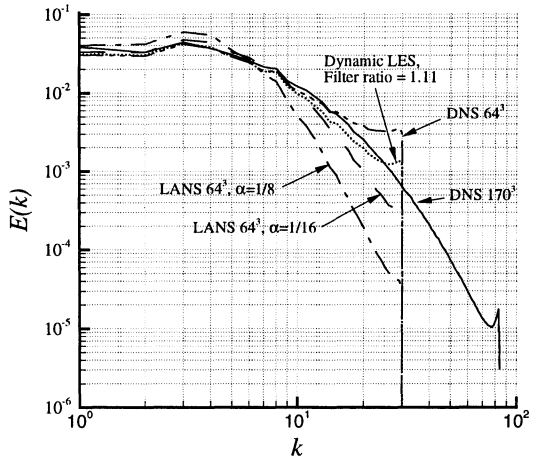


(b)

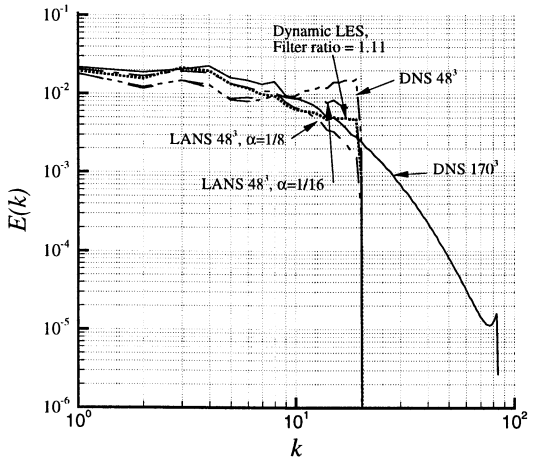
Figure 4: The energy spectra at $t = 2$; (a) $N_\alpha^3 = 48^3$ (b) $N_\alpha^3 = 64^3$.

fore the LANS-α equations can be considered as a nonlinearly dispersive modification of the Navier-Stokes equations. This modification appears in the nonlinearity and depends on the length scale α, and limits the effect of vortex stretching and causes the energy spectrum to fall rapidly for scales smaller than α. The slope of this drop at higher wave-modes can be predicted by an argument similar to Kolmogorov's $k^{-5/3}$ in the inertial range. Prediction of the second slope of the energy spectrum is a topic discussed in Mohseni *et al.* (2001).

Our objective in this study was to investigate the utility of the Lagrangian-Averaged Navier-Stokes (the LANS-α) equations as a subgrid scale model for three-dimensional isotropic decaying turbulence. We performed a Direct Numerical Simulation (DNS) of the Navier-Stokes equations at a Taylor Reynolds number of $\mathrm{Re}_\lambda = 220$ in a periodic box. The initial energy spectrum was peaked at a relatively low wave number (third Fourier mode), making it a suitable test case for testing the LANS-α modeling capabilities.

The only free parameter in the LANS-α simulations is a length scale α which is representative of the spatial scale of the Lagrangian averaging. The LANS-α calculations offer an alternative to Large Eddy Simulation (LES) if a minimum resolution requirement is satisfied. This minimum resolution depends on the initial energy spectrum.

We conclude that for the test case considered in this study the LANS-α equations can capture most of the large scale features of the turbulent flow while the effect of small scales on the large scales were modeled by Lagrangian averaging.



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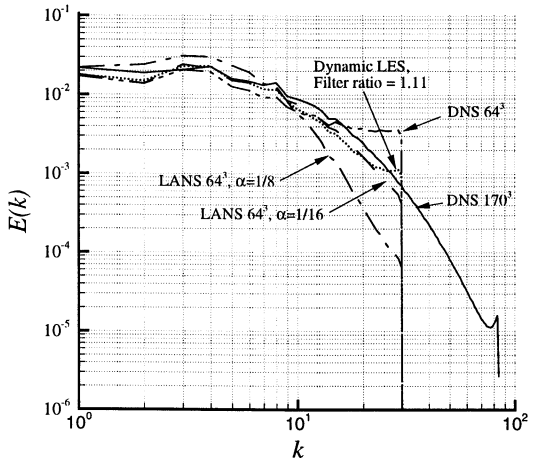

(b)

Figure 5: The energy spectra at $t = 3$; (a) $N_\alpha^3 = 48^3$ (b) $N_\alpha^3 = 64^3$

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