SOME NECESSARY CONDITIONS FOR A NON-LINEAR k - ε MODEL IN CLASSIFIED FLOW PATTERNS WITH A SINGULAR POINT

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ABSTRACT

Realizability conditions of a non-linear k - ε model are investigated firstly for classified mean flow patterns with a singular point. Constraints on the relation between c_μ and the strain and rotation parameters (S, Ω) are derived theoretically for each fundamental flow patterns. Then, distributions of turbulent intensities in a simple flow such as a plane shear layer are derived analytically, using a non-linear k - ε model, and are utilized to identify the model constants in the realizable c_μ -(S , Ω) relation.

INTRODUCTION

It is known that the relation between c_μ and the strain and rotation parameters is sensitive to the prediction of large vortices induced by shear layer instability in turbulent flows (Hosoda, Kimura et al.(1997), Hosoda, Sakurai et al.(1999), Kimura & Hosoda(1999,2000)).

In view of this point, some necessary conditions of a non-linear k - ε model with 2nd and 3rd order terms are investigated for classified 2D & 3D mean flow patterns with a singular point, based on the realizability conditions by Schumann(1977). Fu et al.(1997) investigated the conditions subjected to the constants of Gatski & Spaziale model(1993). In this study, constraints on the relation between c_μ and the strain and rotation parameters (S, Ω) are derived theoretically for each fundamental flow patterns with a singular point. It is also ensured that the relation used by authors to predict large vortices in turbulent flows satisfies these conditions.

Then, to identify the model constants in the realizable c_μ -(S , Ω) relation, distributions of turbulent intensities in a simple flow such as a plane shear layer are derived analytically using a non-linear k - ε model. It is indicated that the method used to derive distributions of turbulent intensities will be applicable to identify the general realizable c_μ -(S , Ω) relation.

CONSTITUTIVE LAW OF NON-LINEAR k - ε MODEL

The constitutive law with quadratic terms can be expressed as follows (Yoshizawa(1984)):

$$-\overline{u_i u_j} = DS_{ij} - \frac{2}{3}k\delta_{ij} - \frac{k}{\varepsilon}D\sum_{\beta=1}^{3} C_\beta\left(S_{\beta ij} - \frac{1}{3}S_{\beta\alpha\alpha}\delta_{ij}\right) \quad (1)$$

where

$$D_t = c_\mu \frac{k^2}{\varepsilon},\ S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i},\ S_{1ij} = \frac{\partial U_i}{\partial x_\gamma}\frac{\partial U_j}{\partial x_\gamma}$$

$$S_{2ij} = \frac{1}{2}\left(\frac{\partial U_\gamma}{\partial x_i}\frac{\partial U_j}{\partial x_\gamma} + \frac{\partial U_\gamma}{\partial x_j}\frac{\partial U_i}{\partial x_\gamma}\right),\ S_{3ij} = \frac{\partial U_\gamma}{\partial x_i}\frac{\partial U_\gamma}{\partial x_j} \quad (2)$$

c_μ is not constant but a function of scalar invariants such as strain parameter, S, and rotation parameter, Ω, defined by

$$S = \frac{k}{\varepsilon}\sqrt{\frac{1}{2}S_{ij}S_{ij}},\ \Omega = \frac{k}{\varepsilon}\sqrt{\frac{1}{2}\Omega_{ij}\Omega_{ij}}, \quad (3)$$

where

$$\Omega_{ij} = \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}. \quad (4)$$

Eq.(1) is equivalent to the following expression Eq.(5) derived by Gatski & Speziale(1993), which are not satisfy the material frame indifference(FMI). It is known that the FMI is not a necessary condition for 3D turbulence (Speziale(1989)).

$$\alpha_1 S_{il}\Omega_{lj} + \alpha_2\left(S_{il}S_{lj} - \frac{1}{3}S_{km}S_{mk}\delta_{ij}\right) + \alpha_3\left(\Omega_{il}\Omega_{lj} - \frac{1}{3}\Omega_{km}\Omega_{mk}\delta_{ij}\right) \quad (5)$$

In the 3rd-order model, the following cubic terms are added to the right side of Eq.(1).

$$-C_4\frac{k^2}{\varepsilon^2}D_t(\Omega_{ik}S_{kl}S_{lj}-S_{ik}S_{kl}\Omega_{lj})$$
$$-C_5\frac{k^2}{\varepsilon^2}D_t(\Omega_{ik}\Omega_{kl}S_{lj}+S_{ik}\Omega_{kl}\Omega_{lj}-\frac{2}{3}S_{mn}\Omega_{no}\Omega_{om}\delta_{ij}) \quad (6)$$

CONSTRAINTS FOR CLASSIFIED FLOW PATTERNS WITH A SINGULAR POINT

Constraints based on the realizability conditions are investigated for a simple shear flow, 2D and 3D mean flow patterns with a singular point, non-linear flows such as a simple accelerated/decelerated shear flow, flow near the longitudinal axis of a swirl jet.

Fundamental 2D and 3D mean flow patterns with a singular point, which can be observed in flows around an obstacle, are classified as shown in Figures 1 and 2(Chong, et al.(1990))..

Simple Shear Flow

A simple shear flow can be expressed by Eq.(7).

$$\begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix}=\begin{pmatrix}0&\alpha&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad (7)$$

where U_i : components of velocity vectors, x_i : spatial coordinates.

If we include the quadratic terms, S_{ij}, Ω_{ij} and S_{ijk} are given by the following matrixes.

[2nd order model]

$$S_{ij}=\begin{pmatrix}0&\alpha&0\\\alpha&0&0\\0&0&0\end{pmatrix},\ \Omega_{ij}=\begin{pmatrix}0&\alpha&0\\-\alpha&0&0\\0&0&0\end{pmatrix}$$

$$S_{1ij}=\begin{pmatrix}\alpha^2&0&0\\0&0&0\\0&0&0\end{pmatrix},\ S_{2ij}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\ S_{3ij}=\begin{pmatrix}0&0&0\\0&\alpha^2&0\\0&0&0\end{pmatrix}$$

$$S_{1ii}=\alpha^2, S_{2ii}=0, S_{3ii}=\alpha^2$$

$$S=\Omega=\frac{k}{\varepsilon}\alpha \quad (8)$$

The Reynolds stresses are described using Eq.(8) as

$$\frac{\overline{u_1u_1}}{k}=\frac{2}{3}+\left(\frac{2}{3}C_1-\frac{1}{3}C_3\right)c_\mu S^2,$$
$$\frac{\overline{u_2u_2}}{k}=\frac{2}{3}+\left(-\frac{1}{3}C_1+\frac{2}{3}C_3\right)c_\mu S^2, \quad (9)$$
$$\frac{\overline{u_1u_2}}{k}=-c_\mu S.$$

Since the realizability inequalities are described as

$$\overline{u_iu_i}>0, \quad (10a)$$
$$\overline{u_iu_i}\cdot\overline{u_ju_j}\geq\overline{u_iu_j}^2, \quad (10b)$$

the constraints on c_μ is reduced to Eq.(11) (Hosoda, Sakurai et al.(1999)). Einstein's summation convention is not used in Eq.(10).

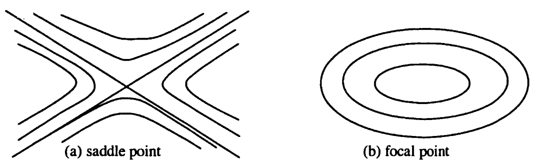

Fig.1 2-D flow patterns with a singular point

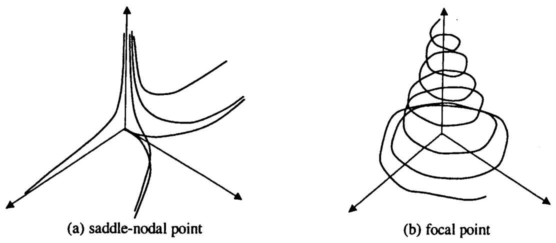

Fig.2 3-D flow patterns with a singular point

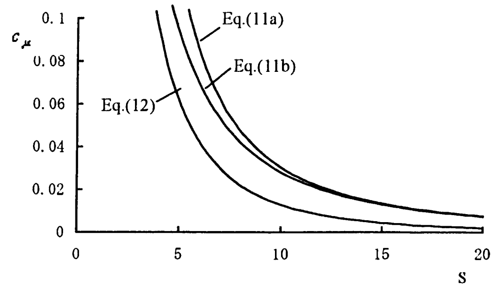

Fig.3 Realizability constraints for a simple shear flow

$$c_\mu<\frac{2}{(C_1-2C_3)S^2} \quad (11a)$$

$$c_\mu<\frac{(C_1+C_3)+\sqrt{(C_1+C_3)^2+4\left\{(2C_1-C_3)(C_1-2C_3)+\frac{9}{S^2}\right\}}}{(2C_1-C_3)(C_1-2C_3)S^2+9} \quad (11b)$$

If we include the cubic terms, Eq.(11b) becomes to

$$c_\mu<\frac{(C_1+C_3)+\sqrt{(C_1+C_3)^2+f_1}}{36(C_4-C_5)^2S^4+\{(2C_1-C_3)(C_1-2C_3)-36(C_4-C_5)\}S^2+9}$$
$$f_1=4\left\{36(C_4-C_5)^2S^2+(2C_1-C_3)(C_1-2C_3)-36(C_4-C_5)+\frac{9}{S^2}\right\} \quad (12)$$

Fig.3 shows Eqs.(11) and (12) with $C_1=0.4$, $C_2=0, C_3=-0.13$, $C_4=-0.02$ and $C_5=0$. Eq.(11b) gives severer condition than Eq(11a) (the condition of non-negative turbulent intensity). Including the cubic terms, Eq.(6) is more restrictive on the realizability.

2-D Flow Patterns with a Singular Point

Fundamental flow patterns with a singular point shown in Fig.1 can be expressed by Eq.(13).

$$\begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix}=\begin{pmatrix}a&\alpha&0\\-\alpha&-a&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad (13)$$

where a and α : positive constants.

S_{ij}, Ω_{ij} and S_{ijk} with the quadratic terms, Eq.(2), are given by the following matrixes.

[2nd order model]

$$S_{ij} = \begin{pmatrix} 2a & 0 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Omega_{ij} = \begin{pmatrix} 0 & 2\alpha & 0 \\ -2\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{1ij} = \begin{pmatrix} a^2+\alpha^2 & -2a\alpha & 0 \\ -2a\alpha & a^2+\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_{2ij} = \begin{pmatrix} a^2-\alpha^2 & 0 & 0 \\ 0 & a^2-\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{3ij} = \begin{pmatrix} a^2+\alpha^2 & 2a\alpha & 0 \\ 2a\alpha & a^2+\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{1ii} = 2(a^2+\alpha^2), S_{2ii} = 2(a^2-\alpha^2), S_{3ii} = 2(a^2+\alpha^2)$$

$$S = 2\frac{k}{\varepsilon}a,\ \Omega = 2\frac{k}{\varepsilon}\alpha \qquad (14)$$

A singular point is classified as follows:

$a > \alpha$: saddle point, $a < \alpha$: focal point

The Reynolds stresses are described as

$$\frac{\overline{u_1u_1}}{k} = -c_\mu S + \frac{2}{3} + \frac{c_\mu}{12}(C_1+C_2+C_3)S^2 + \frac{c_\mu}{12}(C_1-C_2+C_3)\Omega^2$$

$$\frac{\overline{u_2u_2}}{k} = c_\mu S + \frac{2}{3} + \frac{c_\mu}{12}(C_1+C_2+C_3)S^2 + \frac{c_\mu}{12}(C_1-C_2+C_3)\Omega^2$$

$$\frac{\overline{u_3u_3}}{k} = \frac{2}{3} - \frac{c_\mu}{6}(C_1+C_2+C_3)S^2 - \frac{c_\mu}{6}(C_1-C_2+C_3)\Omega^2$$

$$\frac{\overline{u_1u_2}}{k} = \frac{c_\mu}{2}(C_3-C_1)S\Omega \cdot \qquad (15)$$

After some examinations, it is shown that Eq.(16) derived by the Schwartz inequality, Eq.(10b), gives severer condition than the conditions by Eq.(10a).

$$c_\mu = \frac{2}{3}\frac{1}{\left[S\sqrt{1+\frac{1}{4}(C_3-C_1)^2\Omega^2} - f_2\right]} \qquad (16)$$

$$f_2 = \frac{1}{12}\left\{(C_1+C_2+C_3)S^2 + (C_1-C_2+C_3)\Omega^2\right\}$$

Fig.4 shows Eq.(16) for $S = \Omega$.

3-D Flow Patterns with a Singular Point

The similar analysis can be applied to 3-D flow patterns with a singular points shown in Fig.2.

Since the singular points are classified as the saddle-nodal point and the focal point, the derivation of necessary conditions are carried out separately.

Saddle-Nodal Point In the case of saddle-nodal point, the flow is given by

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \qquad (17)$$

The continuity equation is described as the constraint on a,b and c.

$$a+b+c=0 \qquad (18)$$

S_{ij}, Ω_{ij} and S_{ijk} with the quadratic terms are given by the following matrixes.

[2nd order model]

$$S_{ij} = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{pmatrix}, \Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{1ij} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}, S_{2ij} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}, S_{3ij} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

$$S_{1ii} = S_{2ii} = S_{3ii} = a^2+b^2+c^2$$

$$S = \sqrt{2}\frac{k}{\varepsilon}\sqrt{a^2+b^2+c^2}, \Omega = 0, T = \frac{1}{3}\left(\frac{k}{\varepsilon}\right)^3(a^3+b^3+c^3)$$

where $T \equiv (k/\varepsilon)^3 S_{ij}S_{jk}S_{ki}/24$. (19)

One component of turbulent intensities is given by the following equation.

$$\overline{u_2u_2} = -c_\mu\frac{k^2}{\varepsilon}2b + \frac{2}{3}k + c_\mu\frac{k^3}{\varepsilon^2}(C_1+C_2+C_3)\left(-\frac{1}{3}a^2+\frac{2}{3}b^2-\frac{1}{3}c^2\right) \qquad (20)$$

Since $a^3+b^3+c^3 = 3abc$ with $a+b+c=0$, the following relation

$$a^2+b^2+c^2 = \frac{1}{2}\frac{\varepsilon^2}{k^2}S^2$$

can be reduced to

$$x^3+3px+q=0 \qquad (21a)$$

with

$$x = a+b,\ p = -\frac{1}{12}\left(\frac{\varepsilon}{k}\right)^2 S^2,\ q = \left(\frac{\varepsilon}{k}\right)^3 T \qquad (21b)$$

$$u^3 = \frac{1}{2}(-q+\sqrt{q^2+4p^3}),\ v^3 = \frac{1}{2}(-q-\sqrt{q^2+4p^3}). \qquad (21c)$$

$b+c$ and $c+a$ also satisfy Eq.(21).

If the roots of Eq.(21c) are represented by

$$u_0 = |u_0|(\cos\theta_{u_0} + i\sin\theta_{u_0})$$
$$v_0 = |u_0|(\cos\theta_{u_0} - i\sin\theta_{u_0}),$$

the solutions of Eq.(21a) are given by

$$x_1 = 2|u_0|\cos\theta_{u_0},\ |u_0| < x_1 < 2|u_0|$$
$$x_2 = -2|u_0|\cos(\theta_{u_0}-\pi/3),\ -2|u_0| < x_2 < -|u_0| \cdot \qquad (22)$$
$$x_3 = -2|u_0|\cos(\theta_{u_0}+\pi/3),\ -|u_0| < x_3 < |u_0|$$

where

$$\cos 3\theta_{u_0} = \frac{-q}{2\sqrt{-p^3}}, \sin 3\theta_{u_0} = \frac{1}{2\sqrt{-p^3}}\sqrt{|q^2+4p^3|},$$
$$|u_0| = (-p)^{1/2} = \frac{1}{2\sqrt{3}}\left(\frac{\varepsilon}{k}\right)S \qquad (23)$$

Though the range of θ_{u_0} is restricted to

$0 < \theta_{u_0} < \pi/3$, $2\pi/3 < \theta_{u_0} < \pi$, $4\pi/3 < \theta_{u_0} < 5\pi/3$

because of $\sin 3\theta_{u_0} > 0$, it is sufficient to consider one of these ranges.

x_1, x_2, x_3 also satisfy the following constraint due to $a+b+c=0$.

$$x_1+x_2+x_3 = 0 \qquad (24)$$

If we take the following equation as the relation between a,b,c and x_1, x_2, x_3,

$$
\begin{aligned}
&① \; x_1 = a+b, x_2 = b+c, x_3 = c+a \\
&② \; x_1 = b+c, x_2 = c+a, x_3 = a+b \\
&③ \; x_1 = c+a, x_2 = a+b, x_3 = b+c
\end{aligned} \tag{25}
$$

each solutions of Eq.(21a) are given by

$$
\begin{aligned}
&① \; \overline{u_2u_2} = c_\mu \frac{k^3}{\varepsilon^2}(C_1+C_2+C_3)x_3^2 + 2c_\mu\frac{k^2}{\varepsilon}x_3 + \frac{2}{3}k, \\
&\quad -\frac{1}{6}c_\mu(C_1+C_2+C_3)kS^2 \\
&② \; \overline{u_2u_2} = c_\mu \frac{k^3}{\varepsilon^2}(C_1+C_2+C_3)x_2^2 + 2c_\mu\frac{k^2}{\varepsilon}x_2 + \frac{2}{3}k \\
&\quad -\frac{1}{6}c_\mu(C_1+C_2+C_3)kS^2 \\
&③ \; \overline{u_2u_2} = c_\mu \frac{k^3}{\varepsilon^2}(C_1+C_2+C_3)x_1^2 + 2c_\mu\frac{k^2}{\varepsilon}x_1 + \frac{2}{3}k. \\
&\quad -\frac{1}{6}c_\mu(C_1+C_2+C_3)kS^2
\end{aligned} \tag{26}
$$

According to these equations, the condition of non-negative turbulent intensities are reduced to

$$
\begin{aligned}
&-\frac{1}{\sqrt{3}}\frac{\varepsilon}{k}S < x < \frac{1}{\sqrt{3}}\frac{\varepsilon}{k}S: \\
&c_\mu \frac{k^3}{\varepsilon^2}(C_1+C_2+C_3)x^2 + 2c_\mu\frac{k^2}{\varepsilon}x + \frac{2}{3}k \\
&\quad -\frac{1}{6}c_\mu(C_1+C_2+C_3)kS^2 > 0
\end{aligned} \tag{27}
$$

Since the value of Eq.(27) is minimum at

$$
x_{\min} = -\frac{\varepsilon}{k}\frac{1}{C_1+C_2+C_3}, \tag{28}
$$

the constraints are described as

$$
\frac{\sqrt{3}}{C_1+C_2+C_3} < S : c_\mu < \frac{4(C_1+C_2+C_3)}{(C_1+C_2+C_3)^2S^2+6} \tag{29a}
$$

$S < \dfrac{\sqrt{3}}{C_1+C_2+C_3}$:

$$
\begin{aligned}
c_\mu &< \left[\frac{2}{3}\frac{1}{S\left\{\frac{2}{\sqrt{3}} - \frac{1}{6}(C_1+C_2+C_3)S\right\}}\right]_{\min} \\
&= \frac{4}{9}(C_1+C_2+C_3)
\end{aligned} \tag{29b}
$$

Focal Point In the case of focal point, the flow is given by

$$
\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & \alpha \\ 0 & -\alpha & b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{30}
$$

The constraint derived by the continuity equation is

$$
a + 2b = 0. \tag{31}
$$

S_{ij}, Ω_{ij} and S_{ijk} with the quadratic terms are given by the following matrixes.

[2nd order model]

$$
S_{ij} = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2b \end{pmatrix}, \Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\alpha \\ 0 & -2\alpha & 0 \end{pmatrix}
$$

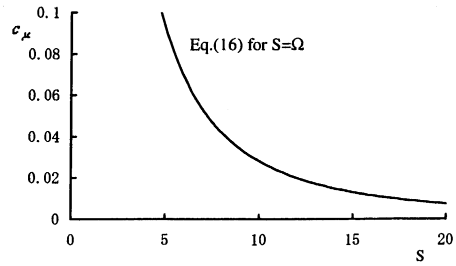

Fig.4 Realizability constraints for 2-D flows

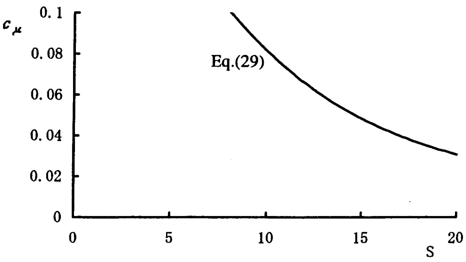

Fig.5 Realizability constraints for 3-D flows with a saddle point

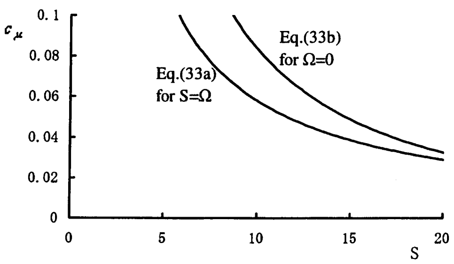

Fig.6 Realizability constraints for 3-D flows with a focal point

$$
S_{1ij} = S_{3ij} = \begin{pmatrix} 4b^2 & 0 & 0 \\ 0 & b^2+\alpha^2 & 0 \\ 0 & 0 & b^2+\alpha^2 \end{pmatrix},
$$

$$
S_{2ij} = \begin{pmatrix} 4b^2 & 0 & 0 \\ 0 & b^2-\alpha^2 & 0 \\ 0 & 0 & b^2-\alpha^2 \end{pmatrix}.
$$

$$
S_{1ii} = S_{3ii} = 6b^2 + 2\alpha^2, \; S_{2ii} = 6b^2 - 2\alpha^2
$$

$$
S = 2\frac{k}{\varepsilon}\sqrt{3b^2}, \Omega = 2\frac{k}{\varepsilon}\sqrt{\alpha^2}, T = -2\left(\frac{k}{\varepsilon}\right)^3 b^3 \tag{32}
$$

The components of Reynolds stress tensors are described as:

$$
\begin{aligned}
\frac{\overline{u_1u_1}}{k} &= -c_\mu\frac{2}{\sqrt{3}}S + \frac{2}{3} + c_\mu(C_1+C_2+C_3)\frac{S^2}{6} \\
&\quad - c_\mu(C_1-C_2+C_3)\frac{\Omega^2}{6}
\end{aligned} \tag{33a}
$$

$$
\begin{aligned}
\frac{\overline{u_2u_2}}{k} = \frac{\overline{u_3u_3}}{k} &= -c_\mu\frac{1}{\sqrt{3}}S + \frac{2}{3} - c_\mu(C_1+C_2+C_3)\frac{S^2}{12} \\
&\quad + c_\mu(C_1-C_2+C_3)\frac{\Omega^2}{12}
\end{aligned} \tag{33b}
$$

$$
\overline{u_1u_2} = \overline{u_2u_3} = \overline{u_1u_3} = 0 \tag{33c}
$$

The constraints on c_μ derived from Eqs.(29), (33a) and (33b) are shown in Fig.5 and Fig.6.

Non-Linear Flows

The constraints of simple non-linear flows are investigated by using the flow model described below:

[simple accelerated/decelerated shear flow]

$$U_1 = b_0 x_1 x_2,\ U_2 = -b_0 \frac{x_2^2}{2} \quad (b_0 : \text{constant})$$

[flow near the longitudinal axis of a swirl jet]

$$U_1 = \frac{a}{x_1}, U_2 = b\frac{x_2}{x_1^2} + \alpha\frac{x_3}{x_1^2}, U_3 = -\alpha\frac{x_2}{x_1^2} + b\frac{x_3}{x_1^2}$$

The results are omitted in this paper because the constraints are not severe compared with ones mentioned above.

DISTRIBUTIONS OF TURBULENT INTENSITIES IN A SIMPLE SHEAR FLOW

The distributions of turbulent intensities in a simple flow such as a plane shear layer are derived analytically using a non-linear k - ε model to identify the model constants in the realizable (c_μ, C_1, C_2, C_3) -(S , Ω) relation.

Simple Shear Flow

The non-dimensional turbulent intensities are

$$\frac{\overline{u_1u_1}}{k} = \frac{2}{3} + c_\mu \frac{2C_1 - C_3}{3} S^2$$
$$\frac{\overline{u_2u_2}}{k} = \frac{2}{3} + c_\mu \frac{2C_3 - C_1}{3} S^2 \qquad (34)$$
$$\frac{\overline{u_3u_3}}{k} = \frac{2}{3} - c_\mu \frac{C_3 + C_1}{3} S^2$$

Anisotropic tensors, m_{11}, m_{22} and m_{33} defined as Eq.(35) are plotted against the ratio of production to dissipation, $c_\mu S^2$, in Fig.7 (dotted line).

$$m_{ij} = \left(\overline{u_i u_j} - (2/3)\delta_{ij} k\right)/k \qquad (35)$$

The agreement with the previous experimental results becomes worse with the increase of $c_\mu S^2$. We therefore introduced following functional forms for $C_1 - C_3$ (Kimura & Hosoda(2000)).

$$C_1 = 0.4 f_S(S), C_2 = 0, C_3 = -0.13 f_S(S)$$
$$f_S(S) = (1 + c_S S^2)^{-1} \qquad (36)$$

The relation between m_{ij} and $c_\mu S^2$ in Eq.(36) with $c_S = 0.02$ is plotted in Fig.7(solid lines). The results in Eq.(36) agree with the experimental results.

Plane Shear Layer

The velocity distribution in a plane shear layer is approximated by the tanh-type distribution shown in Fig.8.

$$U_1 = U_0 \tanh\left(\frac{x_2}{l_0}\right) \qquad (37)$$

Using Taylor expansion, Eq.(37) can be expressed in the vicinity of $x_2 = 0$ as

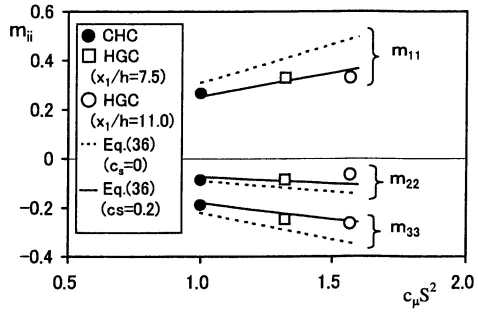

Fig.7 Turbulent intensities in a simple shear flow

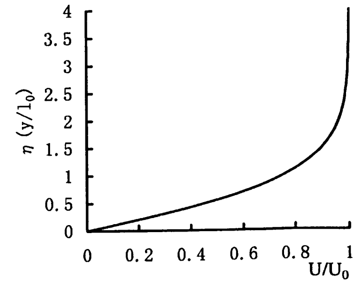

Fig.8 Tanh-type velocity distribution

$$U_1 = a x_2 - b x_2^3,\ a = \frac{U_0}{l_0},\ b = \frac{1}{3}\frac{U_0}{l_0^3}. \qquad (38)$$

To derive the approximate solution of turbulent intensities, c_μ is approximated by Eq.(39) because of $S = \Omega$.

$$c_\mu = c_{\mu_0} \frac{1}{1 + c_D S^2} \qquad (39)$$

The functional forms for k - ε are

$$k = k_0 + k_2 x^2,\ \varepsilon = \varepsilon_0 + \varepsilon_2 x^2. \qquad (40)$$

The non-dimensional expressions on $k_0, k_2, \varepsilon_0, \varepsilon_2$ are

$$k_0 = \frac{a^3}{b} p_0, k_2 = a^2 p_2, \varepsilon_0 = \frac{a^4}{b} q_0, \varepsilon_2 = a^3 q_2 \qquad (41)$$

Substituting Eqs.(38)-(41) to k - ε equations, the relations between p_0, p_2, q_0, q_2 are derived as follows:

$$p_2 = \frac{-\sigma_k \left\{(c_{\mu_0} - c_D) p_0^2 - q_0^2\right\}\left(c_D p_0^2 + q_0^2\right)}{2 c_{\mu_0} p_0^2 \left(p_0^2 + q_0^2\right)} \qquad (42)$$

$$q_2 = \frac{\sigma_\varepsilon f_3}{2 c_{\mu_0} p_0^3 \left(p_0^2 + c_D q_0^2\right)} \qquad (43)$$

$$f_3 = -(c_{\varepsilon_1} c_{\mu_0} - 2 c_D c_{\varepsilon_2}) p_0^2 q_0^3 -$$
$$- c_D (c_{\varepsilon_1} c_{\mu_0} - c_D c_{\varepsilon_2}) p_0^4 q_0 + c_{\varepsilon_2} q_0^5$$

Fig.9 shows an example of distributions of k - ε with $p_0 = 0.04, q_0 = 0.0082, c_{\mu_0} = 0.09, c_D = 0.02,$ $c_{\varepsilon_1} = 1.9, c_{\varepsilon_2} = 2.0, \sigma_k = 1.4, \sigma_\varepsilon = 1.3$. k' and ε' is defined as

$$k' = k\frac{b}{a^3}, \varepsilon' = \varepsilon\frac{b}{a^4}.$$

The numerical value of $p_0 = 0.04$ is evaluated from the experiment by Wygnanski & Fiedler(1969).

The distributions of turbulent intensities can be calculated by using Fig.9. Fig.10 shows the results with $c_S = 0.03$, which should be compared with the experimental results.

CONCLUSIONS

Constraints on the relation between c_μ and the strain and rotation parameters (S, Ω) were derived for some fundamental flow patterns with a singular point. Then, distributions of turbulent intensities in a simple flow such as a plane shear layer were shown as the solutions of a non-linear k - ε model. It was indicated that the method used in this study will be utilized to identify the realizable c_μ -(S , Ω) relation.

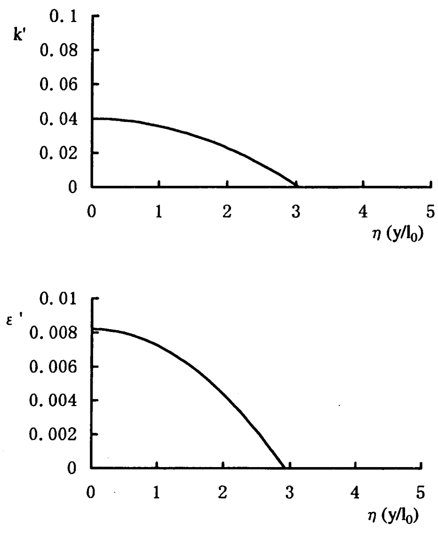

Fig.9 Distributions of k - ε in a plane shear layer

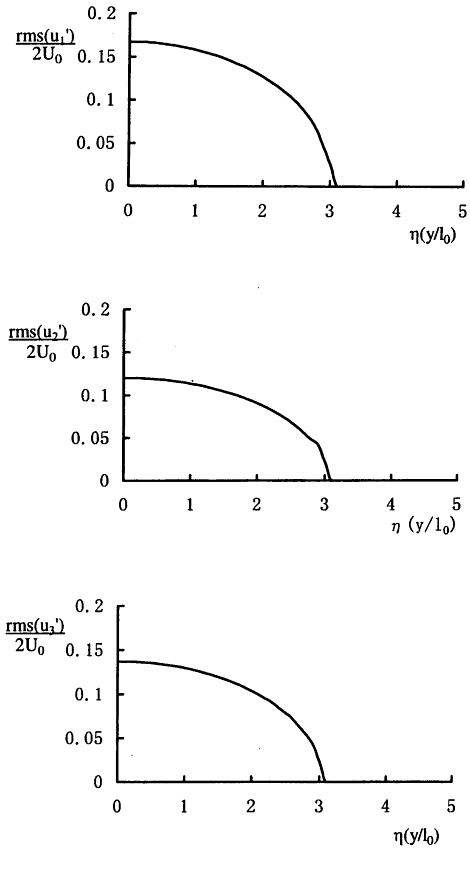

Fig.10 Distributions of turbulent intensities in a plane shear layer

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