INVARIANT ASSUMPTION OF PDF PROFILES IN THE LOG-LAW REGION IN SMOOTH AND ROUGH WALL TURBULENT BOUNDARY LAYERS

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ABSTRACT

The logarithmic velocity profile is considered in zero-pressure-gradient turbulent boundary layers on the rough wall. We have already proposed the new definition of log-law region with using the probability profiles of the stream wise velocity fluctuation for smooth wall. In this report this idea is extended in case of the rough wall one. The measure called Kullback-Leibler divergence is applied for distinguishing the probability profiles, and they are investigated in the overlap region.

INTRODUCTION

We have considered so far the universal velocity distribution in zero-pressure-gradient turbulent boundary layers. Under the invariant assumption of probability density function (abbreviated as PDF hereafter), the log-law velocity profile is derived from the pdf-equation at close to the wall. Our proposal is that the PDF is resemble each other in the log-law region (Tsuji, et al., 1999).

One of the recent interesting topics is the mean velocity distribution in the overlap region in wall-bounded shear flows (Cipra, 1996). The log-law velocity profile is believed to be as firmly established result in turbulence research, however, some researches cast doubts about its existence (Barenblatt & Chorin, 1997). At present, we have no answer to the universal scaling form in the wall bounded shear flows, but even if the log law is a good representation of the experimental data, we still have several questions. Is the slope universal constant? Additive constant is independent of the Reynolds number, isn't it? The researchers use the different values to fit their experimental data. We assume, these disagreements come from the indistinct definition of the log-law region. It is not clear how far the log-law region extends from the wall. Under the process of deriving the log-law profile from the PDF equation (Pope, 1986), we found that the slope should be constant but the additive constant depends on the Reynolds number, which decreases as the Reynolds number increases. The outer edge of the logarithmic region, δ_L, is defined by our invariant assumption of PDF. It is scaled by the boundary layer thickness δ, and the ratio δ/δ_L is depending on the Reynolds number. All these results are for smooth wall boundary layers (Tsuji et al., 1999). The purpose of this paper is to consider whether the invariant assumption of PDF is also satisfied in the rough wall boundary layers, because many experimental works reported that the log-law profile is firmly existing in them.

EXPERIMENTAL CONDIIONS

In a wind tunnel with a test section $0.32 \times 1.06\,\mathrm{m}^2$ in area and 2.6 m in length, a typical two-dimensional turbulent boundary layer is generated. The data are measured at 1900 mm downstream from the leading edge with using I and X-type probes, of which the sensitive region is made of tungsten wire whose diameter is 3.1 μ m and 0.5 mm in length. Also at the seven stations from the leading edge, velocity fluctuation is measured. The probe is operated by a constant temperature anemometer, and the velocity is sampled during 30 sec by 12-bit A/D converter at 10 kHz. The turbulent boundary layer developed over the flat working section on which was attached a sand grain. The height of the roughness elements is $k_s = 1.71$ mm and they are spread all over the bakelite plate as tightly as possible. The flow condition is set at $U_0/\nu = 3.16$, 5.27, 7.38, 9.49, 11.6, $13.71 \times 10^5\,\mathrm{m}^{-1}$, where U_0 is a free-stream mean velocity and ν is a kinematic viscosity. When we consider the unit length, U_0/ν is a kind of Reynolds number. The friction velocity is obtained by the downstream-developing rate of momentum thickness. The error of origin is computed by the method presented by Furuya and Fujita (1976). This is briefly mention in the flowing section.

RESULTS AND DISCUSSIONS

Experimental results are summarized briefly in the following, and the relation between the universal mean velocity profile and the PDF shape is discussed.

Mean Velocity Distribution

The mean velocity distribution on the rough wall boundary layers are represented approximately by the following formula,

$$U^+ = A \cdot \ln(y^+) + B - \Delta U^+ + \frac{2\Pi}{\kappa} w(\zeta) , \qquad (1)$$

where $U^+ = U/u_*$, $y^+ = u_* \cdot (y+\eta)/\nu$, $A = 1/\kappa$ and $\zeta = (y+\eta)/\delta$. While the distance from the top of the sand grain is y , δ is the boundary layer thickness, and u_r is the friction velocity. η is the shift in origin. We believe that κ should be constant while B is slightly depending on the Reynolds number, which are the same for sooth and rough wall boundary layers, here κ is taken to be 0.41. ΔU^+ is the roughness function, which is zero for the smooth wall. The parameter Π determines the strength of the wake function $w(\zeta)$.

Rewriting the above equation in the velocity-defect form, that is, subtracting U^+ from its own value at the edge of boundary layer, U_0, we have the empirical formula with in the overlap region.

$$U_0^+ - U^+ = -A \cdot \ln \frac{(y+\eta)}{\delta_* U_0^+} - E , \qquad (2)$$

where δ_* is the displacement thickness. Hama recommended using the value of $E = 0.6$ (1954), while Furuya & Fujita (1967) suggested that E may depend on the type of roughness elements. In this experiment on the sand grained element, $E \cong 0.9$ for $1920 \le R_\theta \le 10060$ .Equation (2) is expanded as a function of η/δ_* and then fit the experimental data, we obtain the shift origin η (Furuya&Fujira, 1976).

The momentum thickness divided by the distance from the leading edge is plotted as a function of Reynolds number based on the momentum thickness in Fig. 1. This relation is well approximated by the linear fit empirically, therefore the local skin friction is given by the following formula,

$$\frac{c_f}{2} = \frac{d\theta}{dx} = D \cdot R_\theta^{-0.092} \ , \quad D = 8.87 \times 10^{-3} . \qquad (3)$$

Mean velocity profile is plotted in Fig.2(a) for several Reynolds numbers. The log-law profile clearly exists on these rough wall boundary layers. The slope, $A = 1/\kappa$, is almost the same with that of smooth wall.

As the roughness function ΔU^+ is plotted against the k_s^+, it has the slope $1/\kappa$ but is a little larger than the results of sand grained of Furuya & Fujita's experiments, but it is more closer to the relation by Hama's values on wire mesh boundary layers.

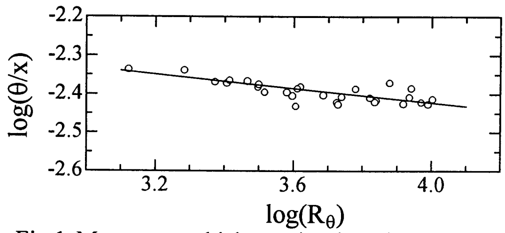

Fig.1 Momentum thickness is plotted as a function of the Reynolds numbers.

The stream wise turbulence intensity, the root mean square, is expressed as u_r, is plotted in Fig. 2(b). Although it is almost constant and decreases close to the wall, on the smooth wall it increases gradually near the wall and its peak is located around $y^+ \approx 15$. This point differs significantly from that of smooth wall boundary layers. The log-law region corresponds to the extent where u_r is convex and gradually decreases outward.

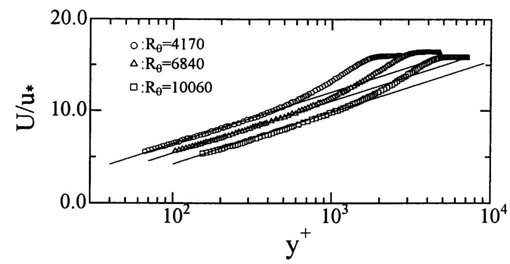

Fig.2(a) Mean velocity profile on the rough wall boundary layers on the rough wall.

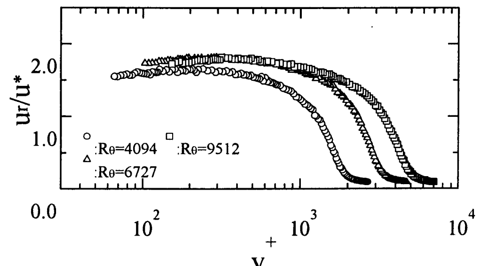

Fig.2(b) The turbulence intensity is normalized by the friction velocity on the rough wall.

The Reynolds stress normalized by u_r is also different from the smooth wall (Fig.2(c)). It is not zero even at the closest point over the rough wall and slightly increase outside. On the smooth wall, however, this value satisfy $-\langle uv \rangle / u_r^2 \approx 0.2$ in the log-law region, and it decreases to zero at the wall.

The key point we mentioned in this section is that the turbulence intensity has a very different profile for smooth and rough wall boundary layers. On the other hand, the mean velocity has the same log-law distribution for these when the appropriate η is adopted on the rough wall one. These are discussed in the next section from the point of PDF profile and the universal scaling.

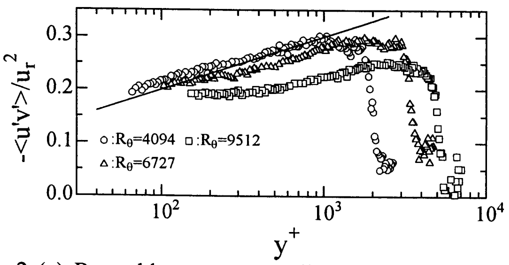

Fig.2 (c) Reynolds stress normalized by the stream wise turbulent intensity on the rough wall.

Invariant Assumption of PDF

The extent of the logarithmic region is classically considered as an equilibrium region where the total shear stress is constant. But the ``equilibrium" is interpreted here as the condition that the PDF profile of the normalized streamwise velocity component does not change. That is, the probability profile remains unique in the log-law region. We call this idea the invariant assumption of PDF.

When the instantaneous velocity in streamwise component is decomposed into mean and fluctuation as $\tilde{u} = U + u'$, we think about the PDF of normalized velocity; u'/u_r, where u_r is r.m.s. value of u'. If the invariant region of the PDF exist, we regard this as the log-law region (Tsuji, et al, 1999). In confirming this assumption, we considered the pdf-equation approximated near the wall region. The PDF is the expected value of the Dirac's delta function in this formula, and it is taken the starting point;

$$P_f(u_p) \equiv \delta(u - u_p) \quad , \quad u = u'/u_r \; , \qquad (4)$$

is defined as the fine grained pdf, where u is a normalized random variable and u_p is a sample variable. And then the PDF is defined as

$$f(u_p) \equiv \langle P_f(u_p) \rangle \; , \qquad (5)$$

where $\langle \; \rangle$ denotes an ensemble average. So the invariant assumption is,

$$\partial f(u_p)/\partial x = 0 \quad , \quad \partial f(u_p)/\partial y = 0 \; , \qquad (6)$$

in log-law region. We have derived the log-law profile from the pdf-equation at close to the wall subject to Eq. (6). The detailed explanation is omitted here, but we adopted the following rational expansion to interpolate the turbulence intensity distribution in the log-law region.

$$u_r^+ = \alpha + \beta_1 (y^+ - \gamma)^{-1} + \beta_2 (y^+ - \gamma)^{-2} + \cdots \; , \qquad (7)$$

The coefficient α means $\alpha = u_r(y^+ \to +\infty)$ and γ is the outer edge of the buffer layer. While the derived log-law formula has the constant slope, the additive constant B depends on Reynolds number. The slope is given as

$$A = C \cdot \alpha \cdot \beta_1 \qquad (8)$$

where C is the function of Reynolds stress,

$$C = -\langle uv \rangle / u_r^2 \; . \qquad (9)$$

We made sure whether Eq. (7) can predict the experimental data on the smooth wall (Tsuji, et al.,1999), and the result was very encouraging. In the logarithmic region Eq. (9) can well interpolate the turbulence intensity in several Reynolds numbers, and it was almost constant; $C \cong 0.2$. The coefficients, α, β_1, were scaled as follows,

$$\alpha = 0.324 \cdot R_\theta^{0.21} \quad , \quad \beta_1 = 86.32 \cdot R_\theta^{-0.21} \; . \qquad (10)$$

Therefore, within the experimental accuracy, the product of α and β_1 is constant independent of the Reynolds number, that is, $\alpha \cdot \beta_1 \cong 28.0$. Then by the Eq. (8), A is constant independent of Reynolds number. About the additive constant B, we are sure from the theoretical procedure that it depends on the Reynolds number.

The above mentioned results are all confirmed on the smooth wall boundary layers. The next step in this report is to check whether the invariant assumption is satisfied on the rough wall boundary layers or not.

First, the streamwise turbulent intensity is considered. We adopted the rational expansion and approximated its profile like Eq. (7). Ordinarily on the smooth wall, u_r^+ has its maximum value around $y^+ \approx 15$ and decrease gradually outward. In the wake region it suddenly goes to zero. Thus, the second term in Eq. (7) is most significant contribution, however the third term has small effect, which is useful to modify the convex profile around the peak of u_r^+. Although the u_r^+ on the rough wall is plotted in Fig.2(b), it is different from that of the smooth wall. The peak, or the local maximum point close to the wall, disappears and then its shape is moderate convex. The closest measurable point is about $y^+ \cong 200$. This location is beyond the buffer layer and is contained in the log-law region in case of the smooth wall. With the wall unit representation, it is not reasonable to compare the rough wall results with the smooth wall, but the physics may be significantly different in the overlap region. The represented formula Eq. (7) is not coming into effect on the rough wall.

Second, the Reynolds stress normalized by u_r,that is C defined by Eq.(9), is required to be constant for the invariant assumption. Townsend defined the structure parameter a_1 as the ratio of the Reynolds shear stress to the turbulent kinetic energy, $a_1 \equiv -\langle u'v' \rangle / [u_r^2 + v_r^2 + w_r^2]$, where v_r and w_r are r.m.s. value of v' and w', respectively. In the inner coordinate system, reported a_1 profiles show a plateau in the vicinity of their peak values which moves outwards with increasing Reynolds numbers. This value is evaluated approximately at $a_1 \cong 0.13$ in the inner region. As the ratio of the fluctuating r.m.s. values are almost constant, $v_r/u_r \cong 0.45$ and $w_r/u_r \cong 0.65$, then $C = a_1 \cdot (1 + 0.45^2 + 0.65^2) \cong 0.2$ is a reasonable estimation. Exactly, it is a weak function of the distance from the wall. We approximated this as

$$C = C_1 + \varepsilon \cdot \log(y^+/p) \quad , \quad p = 10^{C_1/\varepsilon} \quad , \qquad (11)$$

where $C_1 = 0.2$ and $\varepsilon = 0.1$ on the smooth wall boundary layers. In the Fig. 2(c), the solid line represents the Eq. (11). Rough wall has a similar profile but it moves apart as the Reynolds number increase.

To sum up the discussions, we required the empirical formulas Eqs. (7) and (11) for solving the pdf-equation under the invariant assumption , Eq.(6), on the smooth wall. However, the both formulas are not satisfied in case of the rough wall one. So the invariant assumption of PDF is not approved, or it is not an equivalent assumption for the universal log-law velocity distribution. Therefore, the clear mean velocity profile shown in Fig. 2(a) on the rough wall should be comprehend by means of the new concept of turbulence.

Naturally in experimental data analysis, the invariant assumption of PDFs must be a little relaxed. We extract the region where the PDF has a ``similar" profile but not the ``same " one. The Kullback Leibler divergence (KLD) (Kullback, 1959) is used to distinguish the PDF's profile, which is defined as,

$$D(P \parallel Q) \equiv \sum_{\{s\}} P(s) \cdot \log(P(s)/Q(s)) \quad , \qquad (12)$$

where $P(s)$ and $Q(s)$, $\{s\} = s_1, s_2, s_3, \cdots$,are discrete probability distributions. KLD has a non-negative value for any $P(s)$ and $Q(s)$, and it is zero only when $P(s)$ is the same with $Q(s)$. As KLD has a smaller value, then $P(s)$ and $Q(s)$ are more similar with each other. That is, it is a indicator to evaluate how much probability shape resembles.

In order to see the PDF profile more carefully in the log-law region, we compare PDFs with Gaussian distribution. Figure 4 shows the KLD computed at each position from the wall adopting the velocity fluctuation probability and the Gaussian profile. KLD is constant in smooth wall boundary layer within the log-law region. This is consistent with that the probability profile resembles each other, or the invariant assumption is satisfied. On the other hand, in case of rough wall, it is small in the inner region and increases at close to the wall. Comparing with the mean velocity profile in Fig.2(a), within the log-law region, KLD varies significantly from 0.01 to 0.1 . Thus the invariant assumption is not expected on the rough wall.

In the smooth wall boundary layer, the log-law extent is matching the small KLD region, whose value is constant. However the rough wall does not share this character even if there is clear log-law profile in mean velocity. So the question is why the small constant KLD region does not exist in the log-region? We suppose that the mean velocity profile is significantly modified by the quantity of error of origin. As we obtained this quantity by the proper method, the log-law profile is realized. Without adopting this value, we have no log-law profile. The error of origin is the collection of mean velocity profile, but there is no consideration about the fluctuation quantity.

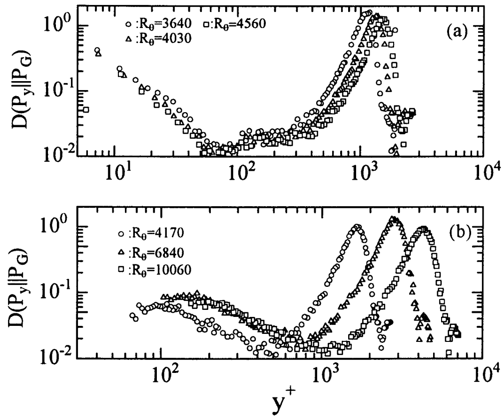

Fig.4 Kullback Leibler divergence (KLD) for the smooth wall (a) and the rough wall (b) boundary layers when $Q(s)$ is adopted as Gaussian in Eq.(12).

SUMMARY

It has been believed so far that the log-law is the universal scaling for any wall bounded shear flows. We though about this problem from the point of PDF of velocity fluctuations. On the smooth wall, the invariant assumption of PDF is equivalent to the log-law distribution, however, this is not approved on the rough wall. The new universal velocity profile is expected on the rough wall boundary layers.

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