THE LAW-OF-THE-WAKE FOR THE REYNOLDS STRESS IN A BOUNDARY LAYER

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ABSTRACT

Experimental data on Reynolds stresses from boundary layers is corrected for Reynolds number effects and correlated as a Wake function. That is the outer stress law minus the common part.

REVIEW OF MEAN VELOCITY PROFILE LAWS

In this paper the two-layer structure of turbulent wall layers is assumed. The transverse coordinate y is measured from the lower wall and the half-height h is the centerline where the velocity is U_O.

Independent variables in the outer and inner regions are

$$Y = \frac{y}{\delta} \qquad y^+ = \frac{u_* y}{\nu} = Y\,\mathrm{Re}_*$$

where $\mathrm{Re}* = \frac{u_*\delta}{\nu} = \frac{u_*}{U_\infty}\mathrm{Re}$

Let the velocity profile be expanded in Poincaré asymptotic expansions. with $\delta_1(\mathrm{Re}_*)$ and $\Delta_1(\mathrm{Re})$ as gauge functions. For the inner region

$$\frac{U(y)}{u_*} = f(y^+, \mathrm{Re}_*) \sim f_0(y^+) + \delta_1(\mathrm{Re}_*)\, f_1(y^+) + \ldots$$

as $\mathrm{Re}* \Rightarrow \infty$

The function $f_0(y^+)$ is the law of the wall. It has been experimentally determined that the same $f_0(y^+)$ applies to pipe flow, channel flow, and boundary layers.

For the outer region

$$\frac{U(y)}{U_O} = F(Y, \mathrm{Re}_*) \sim F_O(Y) + \_(\mathrm{Re}_*)\ F_1(Y) + \ldots$$

as Re* ___

A little analysis (Tennekus and Lumley(1972), Panton (1995)) shows that $\Delta = \frac{u*}{U_O}$, and that the first term is $F_O = 1$. From here on only the first order terms will be retained. Thus, the outer profile can be reorganized,

$$\frac{U(y)}{U_O} = F(Y,\mathrm{Re}) = 1 + \frac{u*}{U_O} F_1(Y)$$

Here F_1 is recognized as the defect law,

$$F_1(Y) = \frac{U(y) - U_O}{u*}$$

Matching between the inner and outer layers requires that

$$F(Y \Rightarrow 0) = \frac{u*}{U_O} f(y^+ \Rightarrow \infty)$$

$$1 + \frac{u*}{U_O} F_{1\ cp}(Y) = \frac{u*}{U_O} f_{o\ cp}(y^+)$$

This produces the overlap laws which are also called the "common parts"

$$f(y^+ \Rightarrow \infty) \equiv f_{o\_cp}(y^+) = \frac{1}{k} \ln y^+ + C_i$$

$$F_1(Y \Rightarrow 0) \equiv F_{1\_cp}(Y) = \frac{1}{k} \ln Y + C_o$$

In principle, constructing a composite expansion forms an expression that is valid for all values of y. An "additive composite" is the sum of the inner and outer expansions minus the common part. For small y the common part cancels the outer function and the inner law gives the correct value. For large y the common part cancels the inner function and the outer is the correct answer. The composite mean velocity is

$$\frac{U(y)}{u*} = f_o(y^+) + \left[F_1(Y) + \frac{U_O}{u*}\right] - \left[F_1(Y) + \frac{U_O}{u*}\right]_{cp}$$

$$= f_o(y^+) + F_1(Y) - [F_1(Y)]_{cp}$$

or

$$\frac{U(y)}{u*} = f_o(y^+) + F_1(Y) - \left[\frac{1}{k} \ln Y + C_o\right]$$

Coles defined the *law of the wake* W(Y) as

$$W(Y) \equiv F_1(Y) - \left[\frac{1}{k} \ln Y + C_o\right]$$

Thus, in the final composite expansion, W(Y) is the outer relation that produces a uniformly valid profile.

$$\frac{U(y)}{u*} = f_o(y^+) + W(Y) \quad \text{where } Y = y^+ / \mathrm{Re}*$$

This expression contains a Reynolds number dependence implicitly. W(Y) is different for pipe flow, channel flow, and boundary layers.

Asymptotic Theory for the Reynolds Stress

The formalism outlined above for the mean velocity can also be applied to the Reynolds stress. Poincaré expansions for the inner and outer regions are,

$$-\overline{uv}\ /u_*^2 = g\,(y^+, Re_*) \sim g_0(y^+) + \ldots$$

as $Re_* \Rightarrow \infty$

$$-\overline{uv}/u_*^2 = G\,(Y, Re_*) \sim G_0(Y)\ + \ldots$$

as $Re_* \Rightarrow \infty$

For channels and round pipes it is known that $G_0(Y) = 1 - Y$.

A differential equation for the inner region connects g_0 and f_0

$$g_0^+ + \frac{df_0}{dy^+} = 1$$

Because f_0 is the same for pipe flow, channel flow and boundary layers, the equation above implies that a single function for g_o will be valid for all situations.

Again, as in the case of the mean velocity, matching produces the overlap laws and common parts. There are simply

$$g(y^+ \_\infty)\ \_\ g\_cp(y^+) = 1$$

$$G(Y \_0)\ \_\ G\_cp\ (Y) = 1$$

Following the pattern established for the mean velocity, one can construct a composite expansion useful for all y. The sum of inner and outer minus the common part is

$$-\overline{uv}\ /u_*^2 = g_0(y^+) + G_0(Y) - G_{0\_cp}$$

$$= g_0(y^+) + G_0(Y) - 1$$

where $Y = y^+ / Re_*$

It is natural to define a law-of-the wake for the Reynolds stress as

$$W_{uv}(Y)\ \_G_O(Y)\ - 1$$

$$-\overline{uv}/u_*^2 = g_0(y^+) + W_{uv}(Y) \qquad (1)$$

Although $G_O(Y)$ and $W_{uv}\ (Y)$ differ by only a constant we will use the wake law in order to emphasize that it is the complementary outer function for the Reynolds stress. An important point is that $W_{uv}\ (Y)$ is part of an expression that is valid for all y and which displays the Reynolds number dependence.

There is a connection between the mean velocity profile and the Reynolds stress in the outer layer. Tennekes and Lumley (1972) show that

$$\frac{dW_{uv}}{d\eta} = -\,\eta\,\frac{dF_1}{d\eta} \quad with \quad \eta \equiv \frac{y}{\Delta} \qquad (2)$$

Recall that F_1 is the velocity defect law. In this equation the symbol Δ is the Rotta-Clauser thickness defined as

$$\Delta \equiv \delta^*\ U_o\ /\ u_*.$$

For pipe and channel flows it is known from theory that $W_{uv}(Y) = -Y$ exactly. The correlation of $W_{uv}(Y)$ data for boundary layer is the subject of this paper.

Universal Inner Reynolds Stress Function $g_0(y^+)$

Substituting with the known outer function for pipes and channel flows, $G_0(Y) = 1 - y^+ / Re_*$, into the composite expansion, Eq. 1 above, and solving for $g_0(y^+)$ gives,

$$g_0(y^+) = \frac{\overline{uv}(y^+)}{u_*^2} + \frac{y^+}{Re_*}$$

This expression allows one to account for Reynolds number effects on the Reynolds stress. It is valid for pipe or channel flows. Experimental data for $\overline{uv}$ $/u_*^2$ has been substituted into the expression above to produce $g_0(y^+)$. Figure 1 shows channel flow results and Fig. 2 pipe flow results. DNS data from channel flow calculations are given in Fig. 3 and data from channel flow calculations comprises Fig. 4.

On each figure a reference curve is given. The reference curve has the equation

$$g_{o\_ref} = \frac{2}{p}\ \arctan(\frac{2k}{p}\ y^+)\ [1 - \exp(\frac{y^+}{C^+})]$$

This relation satisfies the known Taylor's expansion behavior of g_O near the wall, $g_O \sim y^3$ as $y \Rightarrow 0$, and also satisfies the requirement $g_O \sim 1 - \kappa(y^+)$ as $y \Rightarrow \infty$. Two constants in the relation are κ, the von Kármán constant, and an arbitrary scale constant C^+.

The correlation of this data is considered very good. Even at some of the lowest Reynolds numbers one cannot observe any systematic trends.

Reynolds Stress Wake Function for Zero-Pressure-Gradient Boundary Layer

The theme of this paper is to apply a composite expansion for the Reynolds stress to experimental data. In this way one can take account of the Reynolds number dependence (to first order). The logic is as follows. $W_{uv}(Y)$ is known exactly for pipe and channel flows, so the data for $\overline{uv}/u_*^2$ can be used to determine $g_0(y^+)$. Since $g_0(y^+)$ is universal, it can be used to find $W_{uv}(Y)$ for a boundary layer. Solving Eq. 1 for $W_{uv}(Y)$ gives

$$W_{uv}(Y) = -\overline{uv}/u_*^2 - g_0(y^+ = Y\,Re_*) \qquad (3)$$

A curve fit g_{o_ref} represents the universal $g_0(y^+)$ function and experimental data for $\overline{uv}/u_*^2$ used.

Recently, there are two new data sets on boundary layers. Data of Österlund, and Johansson (2001) from Kungl Tekniska Hogskolan (KTH) Stockholm, and of Stanislas (2001) from Ecole Centrale de Lille (ECL) are not yet published. Jens Österlund and Michel Stanislas kindly supplied prepublication copies of their data. However, the authors have noted that corrections for finite X-wire size and wall interference effects have not been applied. In the case of the French data the experimenters have measured a very slight adverse slight pressure gradient. Thus, the reader should view these results as preliminary. The Lille X-wires were 2.5 micron in diameter and 0.5 mm in length while the Stockholm X-wires were 1.27 micron in diameter and 0.3 mm in length. For the Stockholm data this translates into wire length in inner variables of $6.6 < L^+ < 24$. These instruments represent the current state of the art.

Processing the data in accordance with Eq. 2 yielded Figs. 5 and 6. Also shown on the figures is a line marked "Coles." This line was obtained by inserting Coles law-of-the-wake (including the corner correction of Lewkowicz (1982)) into Eq. 3 and integrating.

The first observation is that accounting for Re* effects by subtracting g_o (see Eq. 3) does not improve the correlation of the data. This effect is most prominent near $\eta = 0$ where the data still shows a large scatter. It would appear that data closer to the wall than the peak values are in error. Some of the probe size and wall effect corrections are obviously needed. For $\eta > 0.025$ each data set collapses nicely for all Re* > 2500. However, the sets are not in agreement. The KTH data tends to be below the Coles line and the ECL data tends to be above it. At the present time there is no explanation for the differences.

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Figure 1. Inner Reynolds Stress Function: Channel Flow Experiments

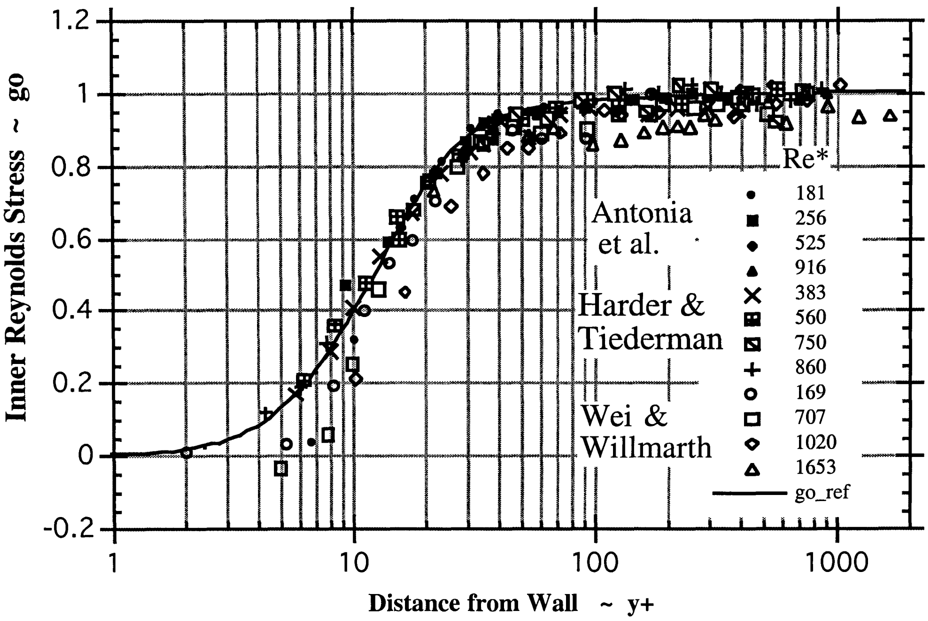


Figure 2. Inner Reynolds Stress Function: Channel Flow Direct Numerical Simulations

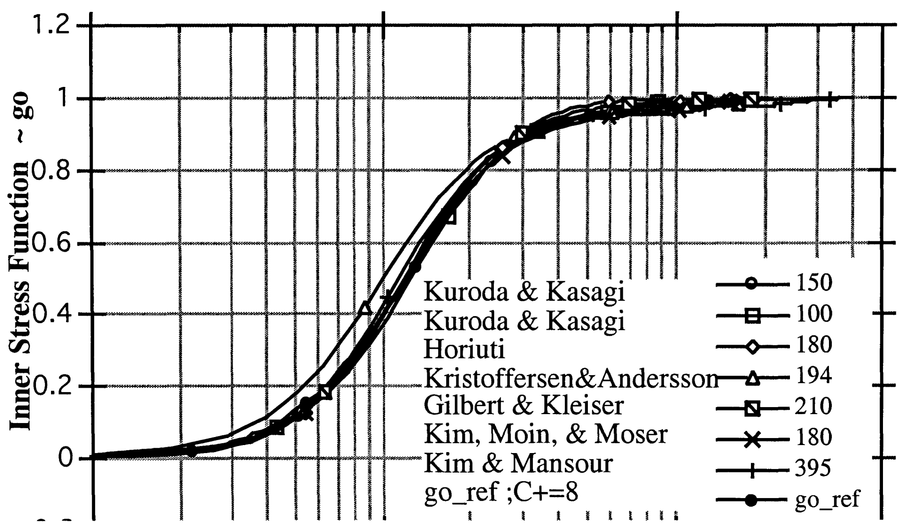


Fig. 3 Inner Reynolds Stress Function:
Pipe Flow Experiments

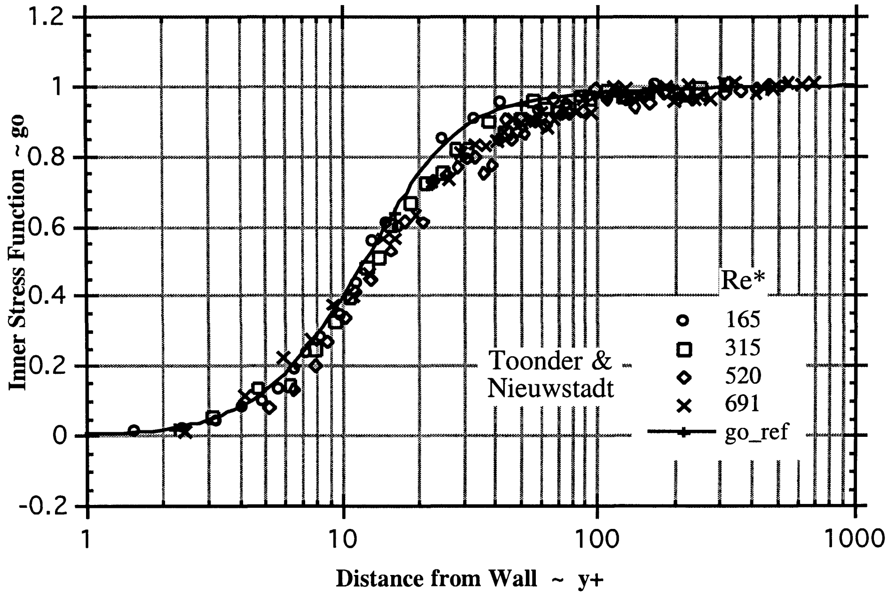
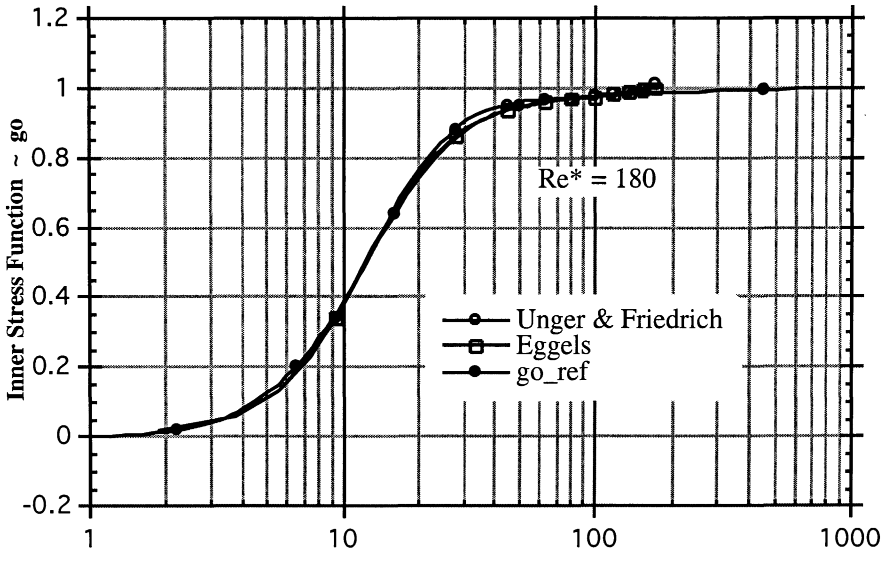
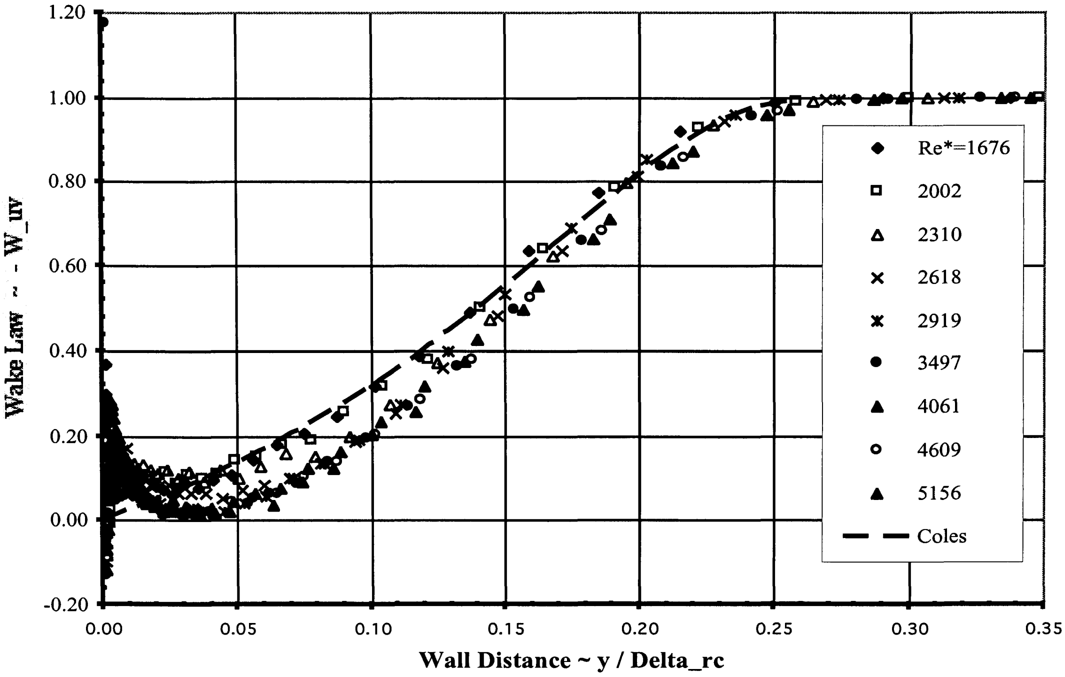


Fig. 4 Inner Reynolds Stress Function;
Pipe Flow, Direct Numerical Simulation



Reynolds Stress ; KTH Stockholm



Reynolds Stress ; Ecole Centrale de Lille