A PRIORI ANALYSIS OF POD REDUCED ORDER MODELS FOR SIMULATION OF TURBULENT FLOWS

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ABSTRACT

Different POD approaches are investigated in a priori tests for turbulent channel flow. In addition to the classical POD decomposition, in which the modes are computed in order to have an optimal representation of the L^2 norm of velocity, POD bases are also obtained by maximizing either the H^1_ϵ norm of velocity or the L^2 norm of vorticity. The velocity fields reconstructed by considering an increasing number of POD modes are compared with those obtained in direct numerical simulation. The accuracy in reproducing statistical moments and near-wall structures and events is evaluated. The physical significance of the POD bases generated in the different approaches is investigated.

INTRODUCTION

In most of the flows of industrial or environmental interest, it is impossible to resolve all the turbulence scales, because of the huge amount of unknowns to be computed. Thus, a challenging problem in numerical simulation of turbulent flows is to devise reduced order models. Such models should imply the computation of a significantly reduced number of unknowns and yet they should reproduce the dynamics of the flow accurately enough. In a broad sense, Reynolds-averaged Navier-Stokes and large-eddy simulations, widely used for turbulent flows, can be considered as reduced order models. However, these approaches lead to simulations still significantly too complex to be used in applications which involve several flow evaluations, such as control or optimization. In this context, low-order models that allow the computational complexity to be further reduced must be used. The basic idea of a class of such methods is to project the solution on a finite-dimension space, characterized by a basis; the number of degrees of freedom is then reduced by considering only a subspace, i.e. a limited number of basis functions.

One of these methods is the proper orthogonal decomposition (POD), in which a set of empirical basis functions is constructed and a Galerkin projection in the POD-function space is employed to derive a simplified dynamical system in the form of a set of nonlinear ordinary differential equations. In the classical approach the POD modes are computed in order to have an optimal representation of the solution in the norm of the Hilbert space L^2 (Lumley, 1967), starting from a data set of flow realizations (snapshots) obtained either through direct numerical simulation or experiments. In order to obtain a low-order model, only the "most important" modes are considered, which in this classical approach may be considered as those mainly contributing to the flow energy. POD techniques have largely been used to analyze signals and/or fields obtained in experiments and simulations for turbulent flows; applications to actual simulation of turbulent flows are still scarce (e.g. Podvin and Lumley (1998) and Omurtag and Sirovich (1999)).

A fundamental aspect to obtain a good compromise between computational complexity and accuracy is the choice of a relevant basis from the physical point of view, in order to be able to identify suitable criteria for the choice of the retained modes. If the flow is characterized by one or more homogeneous directions, each POD mode may be assumed to be a Fourier mode in the homogeneous direction(s). For instance, in turbulent channel flow simulations by Omurtag and Sirovich (1999) the velocity field is represented by a truncated Fourier series in the streamwise and spanwise directions, while POD is used to determine the inhomogeneous basis in the wall-normal direction. In data analysis, this approach can be seen as the application of an inhomogeneous filtering (Adrian et al., 2000). It has been shown by Omurtag and Sirovich (1999) that many degrees of freedom must be retained to obtain an acceptable prediction of mean flow field and second order statistical moments. Moreover, the generation of a POD basis consistent with the Fourier representation of velocity in the homogeneous directions clearly requires to use a wide number of snapshots, and, thus, implies a significant computational cost itself.

In the present paper, we consider POD bases obtained for turbulent channel flow without a priori assuming a Fourier representation in the homogeneous directions and considering a more limited number of snapshots. In this view, POD decomposition is not aimed to extract the most significant information from a large data set, but rather to reorganize a limited amount of data in accordance with a given criterion. In the perspective of the application to actual dynamic simulations, it is useful to investigate the physical meanings of such a decomposition by a priori tests. This type of tests has been used for large-eddy simulation to understand the effects of small scale filtering. In the same way, the comparison between reconstructed POD and DNS fields can give important indications of which kind of physical information is contained in each basis element. This is useful in the perspective of the application not only to the simulation, but also to the analysis of turbulent flow structure. Moreover, we wish to investigate if, in this context (i.e. a limited number of snapshots), POD approaches different from the classical one proposed by Lumley (1967) could be better suited for turbulent wall flows. First, we explore the possibility of using a different norm, namely the norm in the Sobolev space H^1_ϵ, which also accounts for the gradients of the fluid dynamic variables (see also Beux et al., 2000). Second, we propose a POD approach based on vorticity, which gives POD modes ordered in accordance with their contribution to enstrophy.

POD FORMULATION BASED ON THE H^1_ϵ NORM OF VELOCITY

Let us denote the velocity vector of the snapshot p by $U^p = (u_1^p, u_2^p, u_3^p)^T$; the scalar product between two snapshots, U^p and U^q, in H^1_ϵ is defined as:

$$\langle U^p, U^q \rangle_{H^1_\epsilon} = \left\langle \begin{pmatrix} u_1^p \\ u_2^p \\ u_3^p \end{pmatrix}, \begin{pmatrix} u_1^q \\ u_2^q \\ u_3^q \end{pmatrix} \right\rangle + \epsilon \sum_{s=1}^{3} \left\langle \begin{pmatrix} \frac{\partial u_1^p}{\partial x_s} \\ \frac{\partial u_2^p}{\partial x_s} \\ \frac{\partial u_3^p}{\partial x_s} \end{pmatrix}, \begin{pmatrix} \frac{\partial u_1^q}{\partial x_s} \\ \frac{\partial u_2^q}{\partial x_s} \\ \frac{\partial u_3^q}{\partial x_s} \end{pmatrix} \right\rangle \quad (1)$$

where $< \cdot, \cdot >$ denotes the classical scalar product in L^2. Clearly, if ϵ is zero, the L^2 scalar product is recovered from Eq. (1), while for $\epsilon = 1$ we obtain the scalar product in the Sobolev space H^1. Both velocity derivatives and the L^2 scalar product are computed with spectral accuracy, by using the Fourier-Chebyshev representation of U. We refer to Beux et al. (2000) for details. Then, n_s modes can be generated by proper orthogonal decomposition:

$$\phi_p = \sum_{q=1}^{n_s} \alpha_p^q U^q \quad (2)$$

which are mutually orthogonal with respect to the scalar product in H^1_ϵ defined in Eq. (1). The coefficients α_p^q are the components of the p-th eigenvector of the correlation matrix $\mathcal{K}$: $\mathcal{K}_{ij} = \langle U^i, U^j \rangle_{H^1_\epsilon}$.

Since the modes ϕ_p are mutually orthogonal, a generic velocity field U can be reconstructed as follows:

$$U_R = \sum_{j=1}^{n_m} \langle U, \phi_j \rangle_{H^1_\epsilon} \phi_j \quad (3)$$

where n_m is the number of the retained POD modes. Thus, the accuracy of the POD reconstruction depends both on the number of snapshots used n_s and on n_m. By combining

Eqs. (2) and (3), U_R can be rewritten as follows:

$$U_R = \sum_{j=1}^{n_m} \left\{ \sum_{q=1}^{n_s} a_j^q \langle U, U^q \rangle_{H_\epsilon^1} \right\} \phi_j \tag{4}$$

POD FORMULATION BASED ON VORTICITY

Let ω^p be the vorticity field associated to a velocity snapshot U^p, n_s modes can be generated by proper orthogonal decomposition of vorticity:

$$\psi_l = \sum_{i=1}^{n_s} a_l^i \omega^i \tag{5}$$

The coefficients a_l^i are now the components of the l-th eigenvector of the correlation matrix between two vorticity snapshots $\mathcal{K}^\omega$: $\mathcal{K}_{ij}^\omega = \langle \omega^i, \omega^j \rangle_{L^2}$. As previously, both vorticity and L^2 scalar product are computed with spectral accuracy starting from the Fourier-Chebyshev representation of U. The (ψ_l) are mutually orthogonal with respect to the scalar product in L^2, thus, ω^p can be expressed through the POD basis as follows:

$$\omega^p = \sum_{j=1}^{n_s} \langle \omega^p, \psi_j \rangle_{L^2} \psi_j \tag{6}$$

By combining Eqs. (5) and (6), we obtain:

$$\omega^p = \sum_{j=1}^{n_s} A_j^p \sum_{i=1}^{n_s} a_j^i \omega^i \tag{7}$$

in which $A_j^p = \sum_{q=1}^{n_s} a_j^q \langle \omega^p, \omega^q \rangle_{L^2} = \sqrt{\lambda_j} a_j^p$, λ_j being the jth eigenvalue of the correlation matrix. Using now the linearity of the rotational operator, the following equation is obtained:

$$\mathrm{rot}(U^p) = \mathrm{rot}\left(\sum_{j=1}^{n_s} A_j^p \sum_{i=1}^{n_s} a_j^i U^i \right) \tag{8}$$

Thus:

$$U^p = \sum_{j=1}^{n_s} A_j^p \sum_{i=1}^{n_s} a_j^i U^i + S^p \tag{9}$$

where $\mathrm{rot}(S^p) = 0$. However, S^p, which is a linear combination of snapshots, must satisfy the boundary conditions of the problem, i.e. periodic and homogeneous Dirichlet conditions. Thus, necessarily, we obtain that $S^p = 0$.

In conclusion, a generic velocity field U at which corresponds the vorticity field ω can be expressed as follows:

$$U = \sum_{j=1}^{n_m} \sum_{q=1}^{n_s} a_j^q \langle \omega, \omega^q \rangle_{L^2} \theta_j \tag{10}$$

in which $\theta_j = \sum_{i=1}^{n_s} a_j^i U^i$, and correspondingly: $\omega = \sum_{j=1}^{n_m} \sum_{q=1}^{n_s} a_j^q \langle \omega, \omega^q \rangle_{L^2} \psi_j$.

Note that now the POD modes are ordered in accordance with their contribution to enstrophy.

RESULTS AND DISCUSSION

POD bases have been constructed starting from flow snapshots obtained in the direct numerical simulation (DNS) of turbulent channel flow at a Reynolds number, based on the shear velocity and half the channel depth, equal to 108. Details of the simulations can be found in (Soldati and Banerjee, 1998).

Let us consider the POD basis generated by using 20 snapshots and the approach based on the maximization of the H_ϵ^1 norm, with $\epsilon = 0.1$. The snapshots are equally spaced in a time interval roughly corresponding to the characteristic evolution time of a low-speed streak. The velocity fields can be reconstructed as in Eq. (4), by retaining a limited number of modes (n_m). The r.m.s. of the three velocity components and the correlation between streamwise and normal velocity fluctuations, obtained for different values of n_m, are compared to those given by DNS in Figs. 1a and 1b respectively. The considered velocity field is one among the snapshots used for POD basis generation. As expected, if only the first mode is retained the mean flow is completely recovered (not shown here for sake of brevity). More surprisingly, the first mode also gives a significant contribution to the second order statistical moments. However, it is shown in Fig. 1 that 15 modes are required to accurately reproduce the second order statistical moments. Same results have been obtained for flow fields not belonging to the set of snapshots. To better understand those results, it is interesting to compare near wall structures and events surviving in POD reconstructed fields to those present in DNS. This comparison is made for high- and low-speed streaks in Fig. 2 for the same flow field as in Fig. 1. Although significant fluctuations are present also in the first

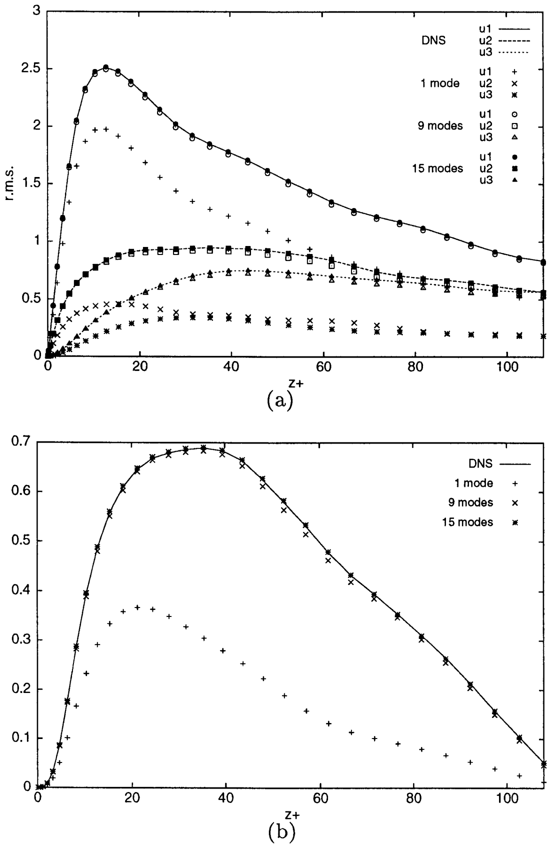


Figure 1: Comparison between DNS and POD reconstructed fields. a) R.m.s. of the velocity components; b) $u'w'$ correlation. z_+ denotes the distance from the wall in the vertical direction, expressed in wall units.

POD mode, which explain its contribution to second order statistical moments, streaks are not captured. However, a rather good representation is already obtained with 5 POD modes. The same analysis can be made for coherent vorticity. Fig. 3 shows a vertical cut of quasi-streamwise vortical structures present in velocity fields reconstructed with different values of n_m and in DNS. The method proposed by Chong et al. (1990) has been employed for the identification of the quasi-streamwise vortices. No coherent vorticity is captured in the first mode (not shown here). It appears from Fig. 3 that at least 15 POD modes are needed to accurately reconstruct vortical structures. As far as sweep and ejections are concerned (not shown here for sake of brevity), an intermediate behavior is observed; indeed, they are well captured with 10 POD modes.

Let us analyze now the spectral behavior of the POD modes. Fig. 4 shows the two dimensional energy spectra of different POD modes on the horizontal plane at the center of the channel. As discussed in the Introduction, if a large number of snapshots is used to generate the POD basis, the POD modes

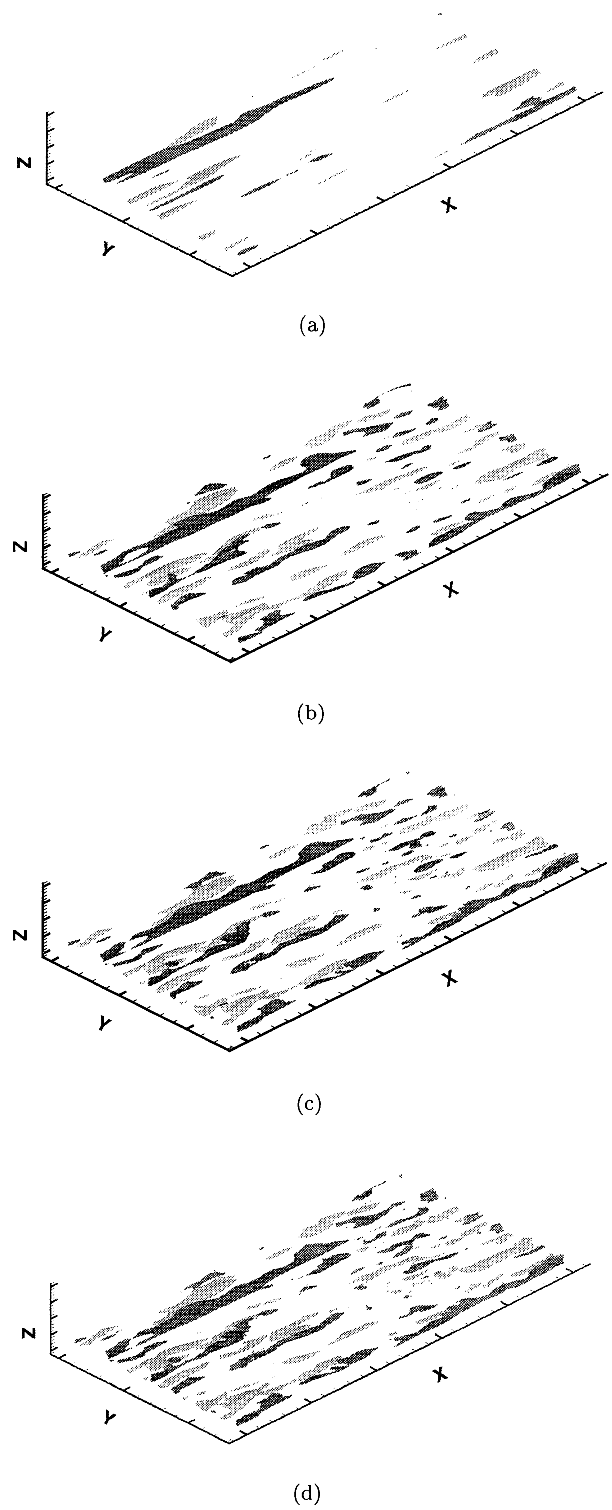
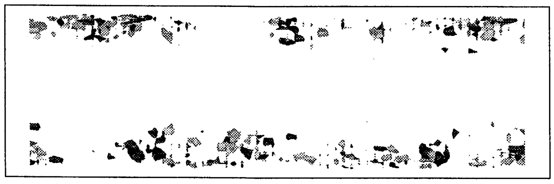


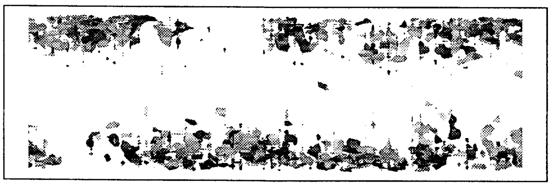
Figure 2: Streaks: iso-contours of $u' = \pm 0.2 U_c$, where U_c is the mean streamwise velocity at the center of the channel. Reconstructed fields with: 1 (a), 5 (b), 10 (c) POD modes; DNS (d).

should coincide with Fourier modes in the homogeneous directions. Of course, this is not obtained in our case.

Indeed, all the POD modes contain significant

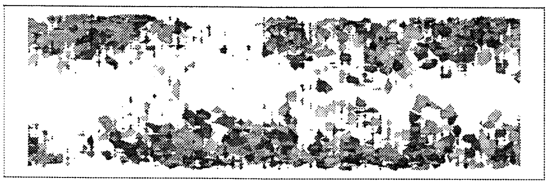


(a)



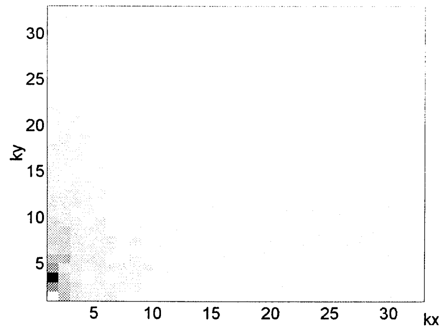
(b)

(c)

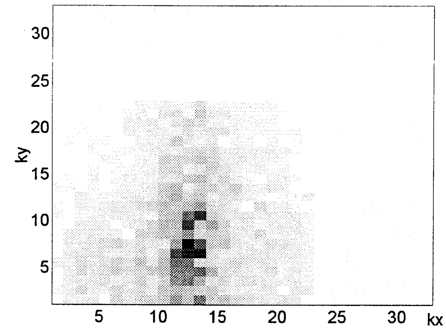


(d)

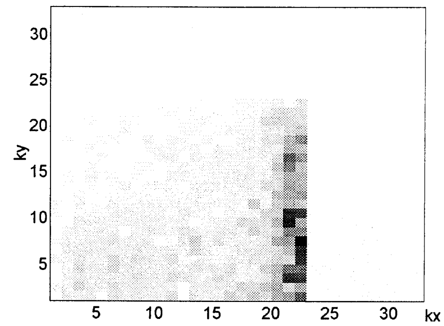
Figure 3: Coherent vorticity. Reconstructed fields with: 5 (a), 10 (b), 15 (c) POD modes; DNS (d).



(a)



(b)



(c)

Figure 4: Two dimensional energy spectra on the horizontal plane at the center of the channel. (a) 1st, (b) 9th and (c) 17th POD modes.

energy on a large interval of waves numbers in the spanwise direction. However, for each mode, energy is mainly contained in a narrow band of wave numbers in the streamwise direction, which moves from low to high wave-number values going from the first to the last POD modes. On homogeneous planes located closer to the wall, the separation of wave numbers containing energy in the different POD modes is less evident also in the streamwise direction. However, it is still true that, as the order of the mode in the POD basis is increased, energy is introduced at higher wave numbers.

Thus, it seems that, by considering an increasing number of POD modes, details, or equivalently small scales, are progressively added in the streamwise direction, while this filtering effect of the POD decomposition is not evident in the spanwise direction.

Finally, let us compare the reconstructed fields for the different POD approaches. The errors in the reconstruction of near wall structures and events are shown for the different approaches in Fig. 5 as a function of n_m. These errors are defined as follows:

$$\mathrm{er}(g) = \frac{1}{M(g)} \frac{\sum_{l \in I(g)} |g(P_l) - g^R(P_l)|}{\max_{l \in I(g)} |g(P_l)|} \tag{11}$$

in which g represents u', $u'w'$ or the coherent vorticity indicator by Chong et al. (1990). Only the nodes which verify the criterion identifying the considered structures or events are

considered in the formula, i.e. the $M(g)$ nodes P_l identified by the set of index $I(g)$.
For the H^1_ϵ formulation, Fig. 5 simply summarizes the information given by the visualizations in Figs. 2 and 3. In particular, it is evident that a larger number of modes is required for vortical structures than for streaks or sweeps and ejections. Except for small differences in the accuracy of the reconstruction of vortical structures, the behavior is practically the same for the different considered formulations. This is due to the fact that the POD modes obtained in the different approaches are very similar as shown, for instance, in Fig. 6, in which the kinetic energy associated to different POD modes, multiplied by the corresponding coefficients and averaged on homogeneous planes, is reported. For the L^2 approach, in which the modes are orthonormal with respect to the L^2 norm, this gives the mean contribution of each mode to the flow energy. Since the bases are very similar, this is roughly true also for the other two formulations. Except for the first mode, which has an energy profile very similar to that of the mean channel flow, the modes are characterized by an energy peak near the wall, that tends to become less localized and to move towards the center of the channel as the mode order increases. Surprisingly, only slight differences are observed between the classical and the other considered approaches. This can be partially justified by the fact that the POD modes obtained in the L^2 formulation are characterized by high shear, especially near the wall, and this also gives a significant contribution to velocity gradients (H^1_ϵ approach) and vorticity. Thus, it seems that the POD approach based on maximization of enstrophy tends to give an optimal representation mainly of vorticity generated by shear, while the reproduction of coherent vorticity remains critical. As stated in the Introduction, we were interested in analyzing POD bases generated with a limited number of snapshots. However, it should be verified whether the above conclusions still hold, at least qualitatively, if the number of snapshots is significantly increased or a larger time interval is considered.

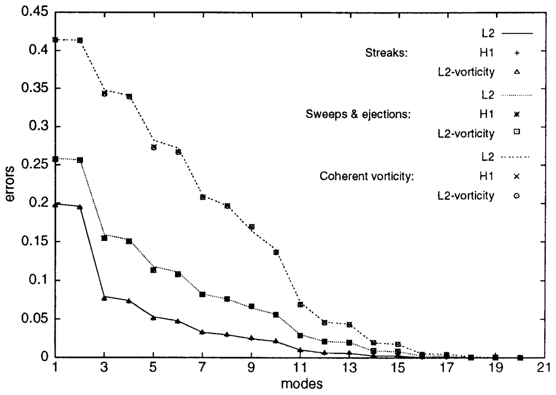


Figure 5: Reconstruction errors. a) Streaks; b) sweeps and ejections; c) coherent vorticity.

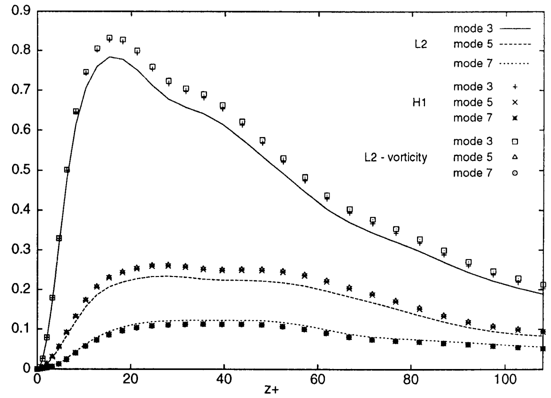


Figure 6: Mean kinetic energy associated to POD modes multiplied by the corresponding coefficients.

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