$\mathcal{H}_\infty$ CONTROL OF LINEAR GLOBAL INSTABILITY IN MODELS OF NON-PARALLEL WAKES

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ABSTRACT

This article investigates the implementation of modern control design techniques on open shear flows using the linear complex Ginzburg-Landau model for the cylinder wake, with the coefficients as scaled by Roussopoulos and Monkewitz (1996). Based on noisy measurements 1.5 diameters downstream of the cylinder, the compensator uses a $\mathcal{H}_\infty$ filter to construct a state estimate which, in turn, is used to compute $\mathcal{H}_\infty$ feedback control at the cylinder to drive the system perturbations to zero. The application of such modern control rules leads to substantially better performance than the proportional measurement feedback proposed by previous studies. Preliminary results on the effectiveness of linear control to stabilize the nonlinear Ginzburg-Landau model are also presented.

POSITION OF THE PROBLEM

The instability and self-sustained oscillation of the flow behind a circular cylinder is a fundamental yet only recently understood problem. Due the numerous engineering consequences of unstable bluff-body flows, the canonical problem of the instability of the cylinder wake has been the focal point of many studies in the past decade (see, e.g., Williamson 1996 for a review). The possibility of controlling this instability, or at least delaying the critical value of Reynolds number characterizing its onset, is an idea that has recently received growing attention (see, e.g., Min and Choi 1999). The growing community of flow control and the interdisciplinary efforts it has accomplished in the last few years makes it now possible to adapt a *control* point of view on *flow* phenomena (see, e.g., Bewley 2001 for survey from this perspective). This paper will investigate the use of linear $\mathcal{H}_\infty$ control theory on a 1D model of the cylinder wake in order to shed some light on some of the central unsolved issues in the control of instabilities in open shear flows.

Open shear flow instabilities, and the major role they play in flow transition, have been analyzed in the last 15 years using the concepts of absolute and convective instabilities (Huerre & Monkewitz 1990). Considering an unstable parallel flow, the instability is called *convective* if a perturbation grows while being advected away by the mean flow and any fixed point in the domain eventually comes back to rest when the upstream disturbance is removed. On the other hand, if the mean advection is not strong enough, the instability will contaminate the entire system and is called *absolute*; in this case, the flow perturbation remains even if the disturbance triggering it is neutralized. These characterizations of parallel flows can be extended to the local analysis of slightly non-parallel flows. Non-parallel flows are often found to contain different regions with different stability characteristics. Flows which are locally convectively unstable everywhere behave as *noise amplifiers*, as they are extremely sensitive to external disturbances, though they are globally stable. On the other hand, flows displaying a sufficiently large pocket of absolute instability behave as *oscillators* (Chomaz, Huerre & Redekopp 1988), and are found to be dominated by a synchronized linear behavior, termed a *linear global mode*. This linear transient will grow in place and eventually saturate due to nonlinearities, leading to self-sustained oscillations such as those in the wake of the circular cylinder.

THE GINZBURG LANDAU MODEL

The system we will consider in this paper, the non-parallel Ginzburg-Landau (GL) system, is the simplest model one can construct that displays a spatial transition from a local convective instability to a local absolute insta-

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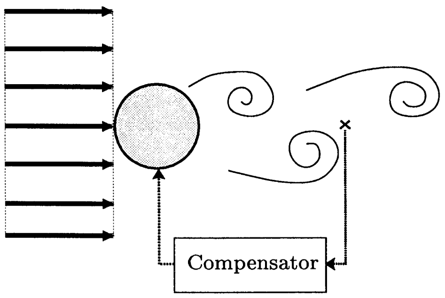

Figure 1: Vortex shedding control approximately modeled by the idealized 1D system: based on noisy measurements downstream of the cylinder, the compensator constructs a state estimate which, in turn, is used to compute control feedback applied at the cylinder itself.

bility. It is given by

$$\frac{\partial \psi}{\partial t} = \left(\mu(x) - U(x)\frac{\partial}{\partial x} + \eta \frac{\partial^2}{\partial x^2}\right)\psi - \lambda|\psi|^2\psi. \tag{1}$$

This equation models the wave amplitude in a spatially extended system and has been used successfully to model the transition of both closed and open flows. In the case of interest here, this equation models quantitatively the Hopf bifurcation which takes place in the cylinder wake at $Re = 47$ and qualitatively the wake behavior as the Reynolds number is increased beyond this value. The present paper focuses primarily on the control and estimation of the linear GL equation (linearizing the system around the solution $\psi = 0$); we will also characterize the efficiency of this linear control on the global nonlinear behavior. We choose complex coefficients in (1) and the dependence of these coefficients on Reynolds number as suggested by Roussopoulos and Monkewitz (1996) in their study of the cylinder wake feedback control problem to facilitate comparison with the existing literature.

THE CONTROL STRATEGY

This work addresses the following idealized model problem: considering the 1D system model (1), what stabilizing effect can be achieved with noisy information about the system 1.5 diameters downstream of the origin and actuation at the origin (i.e., the cylinder location) itself, even if the system is perturbed by unknown external disturbances and significant unmodeled system dynamics? The actuation might be achieved in practice by rotation or transverse oscillation of the cylinder itself; we do not attempt to model accurately the flow actuation in our present 1D analysis, which is focused more on the alteration of the global dynamics in the 1D model of the wake.

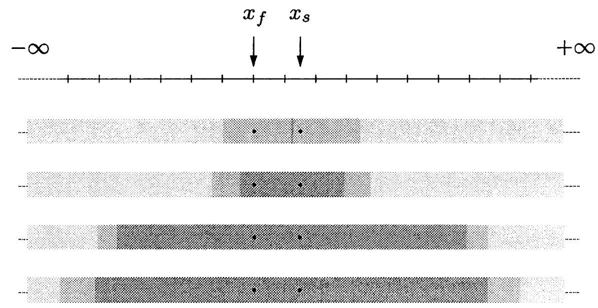

Figure 2: Sketches on the region on the real x-axis of local stability (light grey), convective instability (dark grey) and absolute instability (black) of the uncontrolled GL model for various Reynolds numbers. (a) $Re = 29$, onset of local absolute instability at $x = 1.24$ (b) $Re = 47$, onset of linear global instability, (c) $Re = 235$, (d) $Re = 284$. The positions of the actuator $x_f = 0$ and the sensor $x_s = 1.5$ with respect to the instability zones are also displayed; we seek to determine control forcing, to be applied at x_f, based on the sensor measurement, taken at x_s, in order to stabilize the global dynamics of this system.

We represent the control u as a local forcing at $x = x_f$ on the perturbed GL model and the measurement y of the state at $x = x_s$ in this model such that

$$\frac{\partial \psi}{\partial t} = \mathcal{L}\,\psi + w_\psi(x,t) + \delta(x - x_f)u(t), \tag{2}$$

$$y = \psi(x_s) + w_y(t), \tag{3}$$

As in Roussopoulos & Monkewitz (1996), we take $x_f = 0$ and $x_s = 1.5$. Figure 2 displays the position of both the actuator and the sensor with respect to the local instability zones of the uncontrolled system for a variety of Reynolds numbers.

An appropriate discretization of the continuous GL equation leads to the standard state-space form for the system. Taking α as a free parameter representing the ratio of the strength of the measurement noise to that of the state disturbances, we write this state-space form as

$$\dot{\mathbf{x}} = A\mathbf{x} + \quad B_1\,\mathbf{w} + B_2 u, \tag{4}$$

$$y = C\mathbf{x} + \alpha D_{21}\mathbf{w}, \tag{5}$$

where $\mathbf{x}$ is the state, u the control, and $\mathbf{w}$ the disturbance vector (including both the measurement noise and the state disturbances). The computations presented in this paper have been achieved with a Fourier collocation spectral method for the spatial discretization on a stretched grid around both the sensor and forcing point to ensure localization of the sensing and forcing. The control design applied by our study is the linear $\mathcal{H}_\infty$ control approach introduced by Doyle, Glover, Khargonekar and Francis (1989). This control methodology can be briefly described as the following: (1) choice of a quadratic cost function, (2) choice of the design parameters, and (3) computation of the control matrices. The performance of the closed-loop plant depends strongly on the several decisions made at each of these steps. The cost function $\mathcal{J}$ must weigh together the state $\mathbf{x}$, the control effort u, and the noise $\mathbf{w}$; moreover, since the GL operator is time invariant, one can apply the control theory for infinite time horizons, which leads to the following general form for the cost function:

$$\mathcal{J} = \int_0^\infty \left[\mathbf{x}^* Q \mathbf{x} + \ell^2 u^* R u - \gamma^2 \mathbf{w}^* S \mathbf{w} \right] dt. \quad (6)$$

where Q, R and S are positive definite matrices. The $\mathcal{H}_\infty$ control approach allows one to compute the control u that minimizes the cost function in the presence of the "worst-case" disturbance $\mathbf{w}$ that simultaneously maximizes the cost function, in the spirit of a noncooperative game or saddle-point problem. More detailed review of the $\mathcal{H}_\infty$ control design procedure is given in Bewley (2001).

In addition to α, the two other design parameters are the weighting on the control penalty, ℓ (large ℓ resulting in small control amplitude) and the weighting on the disturbance penalty, γ (large γ resulting in small "worst-case" disturbance amplitude accounted for during the controller design). A case of particular interest is the $\mathcal{H}_2$ (or "optimal control") approach, which is achieved by taking the $\gamma \to \infty$ limit, resulting in a worst-case disturbance of vanishing amplitude to be accounted for during the controller design. It may be shown that the control design procedure in this limit is essentially equivalent to the control design that minimizes the cost function under a white-noise assumption for $\mathbf{w}$.

PERFORMANCE ANALYSIS

In the previous section, we briefly discussed the reformulation of the GL system into standard state-space form and the design an $\mathcal{H}_\infty$ compensator for this system with three design parameters ℓ, α and γ. This section now examines some of the relevant questions concerning the effectiveness of this compensation on both the linear and the nonlinear GL system.

Linear control of the linear GL equation

We now introduce three appropriate measures of performances for the present problem. The first measure is the maximum Reynolds number for stability of the closed-loop system.

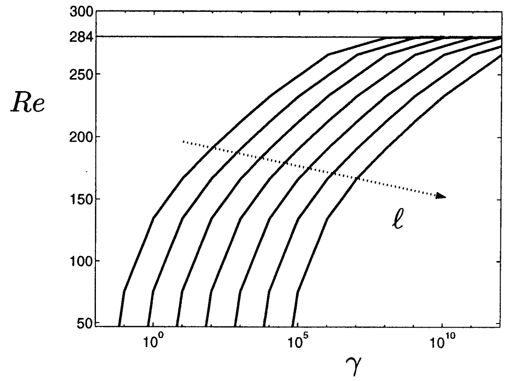

Figure 3: Maximum Reynolds number for stability of the controlled GL system with full-information $\mathcal{H}_\infty$ control as a function of the robustness parameter γ for various values of the control penalty ℓ. The horizontal line represents the $\mathcal{H}_2$ limit $Re = 284$ (independant of ℓ).

If one does not apply control, the system is unstable as soon as Re exceeds the threshold value of 47. The higher the new threshold for instability is, the more effective the control is for *delaying transition.* The other two performance measures are the transfer function 2-norm $||T_{\mathbf{xw}}||_2$, quantifying the *amplification of zero-mean white Gaussian disturbances* by the closed-loop system, and the transfer function infinity-norm $||T_{\mathbf{xw}}||_\infty$, quantifying the *amplification of disturbances with "worst-case" structure* by the closed-loop system. Broadly speaking, the transfer-function norms $||T_{\mathbf{xw}}||_2$ and $||T_{\mathbf{xw}}||_\infty$ represent how the wake model with control feedback applied responds to benign and malevolent disturbance respectively. As a consequence, $||T_{\mathbf{xw}}||_2$ for a given stable system is always smaller than $||T_{\mathbf{xw}}||_\infty$.

The optimal control with full information stabilizes the wake model up to a Reynolds number of 284, which corresponds to stabilization of 7 linear global modes. This is the *stabilizability limit* of the model system with the chosen actuator. No control design is able to stabilize the system at a higher Reynolds number (one reason this approach is termed "optimal" control). The estimator itself is able to fully recover the state from noisy measurements at $x_s = 1.5$ up to a Reynolds number of 235; this is the *detectability limit* of the model with the chosen sensor. Due to the separation principle between the control and estimation problems in the $\mathcal{H}_2$ framework, the compensator formed by combining the estimator and the controller will stabilize the plant up the minimum of these two values, $Re = 235$. This number, corresponding to the stabilization of 6 linear global modes, compares quite favorably with the maximum Reynolds number $Re = 64$

$\log\gamma$	case 1 *Re*	case 2 *Re*	case 3 *Re*
1	47.0	47.0	47.0
2	50.9	47.0	47.0
3	99.4	47.0	47.0
4	129.6	89.1	94.2
5	151.2	122.5	120.0
6	169.7	142.7	140.7
7	186.5	163.7	160.6
8	218.4	201.3	197.5
9	231.5	218.4	215.5
10	235.0	221.6	231.6
∞	235.0	235.0	235.0

Table 1: New stability Reynolds number for estimator-based $\mathcal{H}_\infty$ control of the linear GL equation as a function of the robustness parameter γ. Case 1: $\ell = 1, \alpha = 1$; case 2: $\ell = 1, \alpha = 100$; case 3: $\ell = 100, \alpha = 1$.

α	Control	$\|\|T_{\mathbf{xw}}\|\|_2$	$\|\|T_{\mathbf{xw}}\|\|_\infty$
0.01	RM96 (proportional)	55.8	653
	$\mathcal{H}_2$ ($\ell = 10000$)	21.0	130
	$\mathcal{H}_2$ ($\ell = 100$)	20.3	122
	$\mathcal{H}_2$ ($\ell = 10$)	12.1	44.1
	$\mathcal{H}_2$ ($\ell = 1$)	8.0	18.0
	$\mathcal{H}_2$ ($\ell = 0.01$)	7.6	15.0
100	RM96 (proportional)	212	2500
	$\mathcal{H}_2$ ($\ell = 10000$)	165	1343
	$\mathcal{H}_2$ ($\ell = 100$)	161	1308
	$\mathcal{H}_2$ ($\ell = 10$)	121	796
	$\mathcal{H}_2$ ($\ell = 1$)	99.4	582
	$\mathcal{H}_2$ ($\ell = 0.01$)	96.8	553

Table 2: Comparison of transfer function norms $||T_{\mathbf{xw}}||_2$ and $||T_{\mathbf{xw}}||_\infty$ at $Re = 60$ for six types of control: the proportional strategy of Roussopoulos and Monkewitz (1996), measurement-based $\mathcal{H}_2$ control for $\ell = 10000$, $\ell = 100$, $\ell = 10$, $\ell = 1$ and $\ell = 0.01$. Top: $\alpha = 0.01$, bottom: $\alpha = 100$.

which could be stabilized by the proportional control approach developed by Roussopoulos and Monkewitz (1996), which stabilized only one linear global mode. Our first conclusion is therefore that the optimal control design is much more effective in delaying system instability than simpler control strategies. A "robust" control design can also be developed with this approach, either for the full-information case or for the measurement-based compensator. Figure 3 displays the variations of the *Re* limit for the full information $\mathcal{H}_\infty$ control for various values of the control penalty ℓ. Table 1 extends these results to the $\mathcal{H}_\infty$ compensator, *i.e.* to the case in which the controller does not have access to full information but instead constructs a state-estimate based on the measurement obtained by the sensor.

The results from Figure 3 and Table 1 allow us to draw 3 main conclusions. It appears first that the maximum stabilized Reynolds number depends monotonically on the robustness parameter γ before reaching an asymptotic value. These asymptotic values as γ approaches infinity are the values given by the optimal control approach ($Re = 284$ for the full-information case in Figure 3, $Re = 235$ for the estimator-based case in Table 1), which was expected as the $\mathcal{H}_2$ control design reduces exaclty to the $\mathcal{H}_\infty$ control design for $\gamma = \infty$. Another important conclusion is that, for a given robustness parameter γ, increasing the penalty on the control ℓ or the measurement noise strength α seems to result in decreasing the maximum *Re* which is stabilized. We see therefore that the $\mathcal{H}_\infty$ approach is not more efficient than the $\mathcal{H}_2$ control in delaying the instability. This last result can be easily interpeted since the local minimum for $\mathcal{J}$ given by the saddle point computation when $\gamma < \infty$ is necessarily larger than (or equal to) the global minimum given by the optimal control when $\gamma \to \infty$; introduction of the disturbance effectively *detunes* the optimal compensator.

Another advantage of the modern control design over the simpler proportional scheme of Roussopoulos & Monkewitz (1996) is the decrease in the transfer function norms. Table 2 presents a comparison at $Re = 60$ for 2 different noise strengths between the values of the transfer function 2-norms and infinity-norms for the RM96 proportional approach and for the present optimal control approach with various values for ℓ. A first result to be observed is the monotonic dependance of the transfer function norms on the control penalty ℓ and the measurement noise to state disturbance ratio α: for a given α, increasing ℓ results in less authority of the control and therefore a worse disturbance rejection (larger values for the transfer function norms); for a given ℓ, increasing α results in having less reliable measurements and therefore again a worse disturbance rejection. By analysing the results of Table 2, it is clear that, both in the case of low ($\alpha = 0.01$) and high ($\alpha = 100$) noise strength, applying modern control on the present system is more effective than proportional control in terms of disturbance rejection by the closed-loop system, whether it is rejection of "white" disturbances (lower value of $||T_{\mathbf{xw}}||_2$) or rejection of "worst-case" disturbances (lower value of $||T_{\mathbf{xw}}||_\infty$). Therefore, even in the domain where the simple proportional control of RM96 stabilizes the model, it is preferable to apply modern control.

A final important aspect to be considered in this linear study is the relative performance of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controls. Table 3 presents the values of the two transfer function norms for various Reynolds numbers with the two control strategies applied: the $\mathcal{H}_2$ control and the

	Re	$\|\|T_{\mathbf{xw}}\|\|_2$	$\|\|T_{\mathbf{xw}}\|\|_\infty$	Δ_2	Δ_∞
	60	8.5	20.0	-	-
$\mathcal{H}_2$	100	40.9	224.8	-	-
	150	5541	17119	-	-
	200	5454553	13076172	-	-
	60	10.3	11.7	+21.2%	-41.5%
$\mathcal{H}_\infty$	100	70.3	103.6	+71.9%	-53.9%
	150	6385	8594	+15.2%	-49.8%
	200	5465686	6625000	+0.2%	-49.3%

Table 3: Comparison of transfer function norms $||T_{\mathbf{xw}}||_2$ and $||T_{\mathbf{xw}}||_\infty$ between two types of control for $\ell = 1$ and $\alpha = 1$: $\mathcal{H}_2$ control and $\mathcal{H}_\infty$ control with the smallest value possible for γ. Δ_2 is the relative difference between the transfer function 2-norms and Δ_∞ the relative difference between the transfer function ∞-norms.

$\mathcal{H}_\infty$ control with the smallest possible value for γ (termed γ_0). These computations were achieved with moderate values of both the control penalty ($\ell = 1$) and the measurement noise to state disturbance ratio ($\alpha = 1$). Important conclusions can be drawn from the results displayed on the Table 3. It appears first that both transfer function norms increase monotonically as the Reynolds number is increased, indicating hightened sensitivity of the closed-loop system to disturbances as the number of linear global modes increases. It appears also that applying $\mathcal{H}_\infty$ control instead of $\mathcal{H}_2$ control results in an increase in $||T_{\mathbf{xw}}||_2$ and a decrease in $||T_{\mathbf{xw}}||_\infty$. As a consequence, a design trade-off should be considered between white disturbance rejection and worst-case disturbance rejection. It also apparent that, for increasing Re, it is preferable to apply $\mathcal{H}_\infty$ control than a $\mathcal{H}_2$ control, as the $\mathcal{H}_\infty$ approach gives a very large decrease in $||T_{\mathbf{xw}}||_\infty$, denoted Δ_∞, while giving only a very small increase in $||T_{\mathbf{xw}}||_2$, denoted Δ_2. We thus conclude that applying a robust control as one approaches the stabilizability limit of the system, $Re = 235$, presents a substantial advantage over an optimal control.

Linear control of the nonlinear GL equation

The idea of applying the linear control strategy to the nonlinear GL model is appealing for two reasons: first, because having designed and computed a control, it is straightforward to test it on the nonlinear equation, and second, because the nonlinear model better addresses the *real* problem of interest, that is, stabilizing the nonlinear synchronized behavior. Figure 4 displays a simulation for $Re = 100$ with random initial condition for the simulation when the optimal control is applied at time $t = 150$. This simulation was performed with a semi-implicit Adams-Bashforth-Crank-Nicholson (ABCN) time advancement. It can

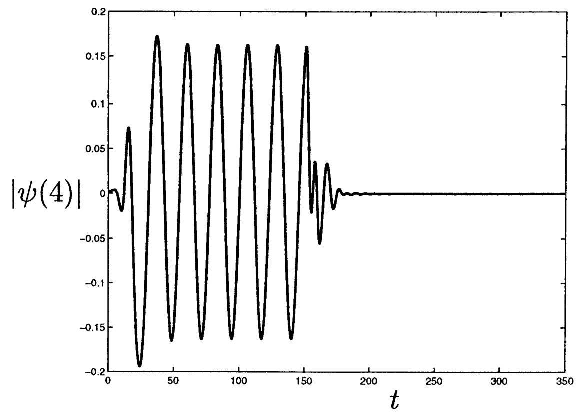

Figure 4: Linear control of the nonlinear GL equation for $Re = 100$: time evolution of the amplitude 4 diameters downstream of the cylinder. The $\mathcal{H}_2$ control is switched on at $t = 150$ and drives the oscillating state to zero.

be seen in Figure 4 that linear control effectively stabilizes the nonlinear system and drives the state to zero extremely quickly. Further computations will be needed to explore this result at different Reynolds numbers and for various different control strategies, though this preliminary result is quite encouraging in this regard.

CONCLUSION

This paper investigates the use of linear $\mathcal{H}_\infty$ control theory on a simple model of the cylinder wake to answer some fundamental unanswered questions regarding the control of open shear flows instabilities. It was shown that the application of such modern control rules leads to substantially better performance than the proportional measurement feedback proposed by previous studies by delaying the Reynolds number for onset of linear global instability by a factor of 5 and significantly decreasing the sensitivity of the system to external perturbations. The advantage of using *robust* over *optimal* control was shown to be of particular importance near the stabilizability limit of the system, and preliminary results were given where the *linear* control stabilized the entire *nonlinear* Ginzburg-Landau equation.

One of the conclusions from Monkewitz (1989) and Huerre & Monkewitz (1990) concerning control of open flows was that it was very likely that each linear global mode needed to be stabilized by a separate actuator/sensor pair. The present paper has shown that, with the proper control algorithm, this is in fact not the case. The present control strategy stabilizes 6 linear global modes with a single actuator/sensor pair, which is in fact the "stabilizability limit" of the present system - no other control strategy can stabilize this linear system further with the actuator and sensor configured in the present manner.

Significant fundamental questions still remain unanswered:

(1) What is the best position for the actuator and the sensor to obtain an *overall* best performance, and what is the new maximum Reynolds number and number of global modes which can be stabilized?

(2) How is the "stabilizability limit" of the system characterized in terms of its eigenmodes? What is the limiting factor preventing stabilization at higher Re?

(3) Under what conditions is the linear control effective on the nonlinear equation in the synchronized, self-sustained, limit-cycling behavior? What is the effect of the noncooperative aspect of the "robust" formulation on this problem?

(4) What filtering technique is most appropriate for estimation of the nonlinear equation?

(5) Do control strategies designed for one Reynolds number work well at other Reynolds numbers?

These questions are currently under active investigation by the authors, and will be reported in a future paper.

Acknowledgements

The authors gratefully acknowledge the encouragements and technical advice of Patrick Huerre (Directeur-State) and Jean-Marc Chomaz (Directeur-Adjoint) at LadHyX. The first author acknowledges the funding of CTE, Ecole des Mines de Paris.

References

Bewley, T.R. 2001, "Flow control: new challenges for a new renaissance", *Progress in Aerospace Sciences*, **37**, 21-58

Chomaz, J.M., Huerre, P. and Redekopp L.G., 1988, "Bifurcations to local and global modes in spatially developing flows", *Phys. Rev. Lett.* **60** (1), 25-28

Doyle, J.C., Glover, K., Khargonekar, P.P. and Francis, B.A., 1989, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems" *IEEE Trans. Aut. Cont.* **34**, 831-847

Huerre, P. and Monkewitz, P.A., 1990, Local and global instabilities in spatially developing flows *Annu. Rev. Fluid. Mech.* **22**, 473-537

Monkewitz, P.A., 1989, "Feedback control of global oscillations in fluid systems" *AIAA Paper* No 89-0991

Min, C. and Choi, H., 1999, "Suboptimal feedback control of vortex shedding at low Reynolds numbers" *J. Fluid Mech.* **401**, 123-156

Roussopoulos, K. and Monkewitz, P.A., 1996, "Nonlinear modelling of vortex shedding control in cylinder wakes" *Physica D* **97**, 264-273

Williamson, C.H.K., 1996, "Vortex dynamics in the cylinder wake" *Annu. Rev. Fluid. Mech.* **28**, 477-539