MODELLING OF STREAMLINE CURVATURE EFFECTS ON TURBULENCE IN EXPLICIT ALGEBRAIC REYNOLDS STRESS TURBULENCE MODELS

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ABSTRACT

A curvature correction for explicit algebraic Reynolds stress models (EARSM), based on an approximation of the formal derivation of the weak-equilibrium assumption in a streamline-based co-ordinate system is presented. The proposed correction is (i) Galilean invariant, (ii) identical to the formal transformation if the magnitude of the velocity along the streamline is constant and (iii) vanishes in flows without curvature/rotational effects. The importance of the curvature correction in EARSMs have been shown for rotating homogeneous shear and rotating channel flows where the proposed model performs well.

INTRODUCTION

Turbulent flows over curved surfaces, near stagnation and separation points, in vortices and turbulent flows in rotating frames of reference are all affected by streamline curvature effects. Strong curvature and/or rotational effects form a major cornerstone problem also at the Reynolds stress transport modelling level, and pressure-strain rate models that are able to accurately capture rapidly rotating turbulence are rare. In more moderate situations the SSG model (Speziale, Sarkar & Gatski 1991), and derivations thereof, show rather good behaviour in rotating flows such as rotating homogeneous shear flows, see Gatski & Speziale (1993). Standard eddy-viscosity models fail in describing effects of local as well as global rotation.

Algebraic Reynolds stress models, Rodi (1972, 1976), are the results of applying the weak equilibrium assumption on the full differential models. In the weak equilibrium limit of turbulence, the Reynolds stress anisotropy tensor, $a_{ij} \equiv \overline{u_i u_j}/K - (2/3)\delta_{ij}$, is assumed to be constant following a streamline. Neglecting also the diffusion of the anisotropy tensor results in an implicit purely algebraic relation for a_{ij}. Algebraic modelling has had a renewal during the last decade after it was found that the resulting implicit algebraic relation for a_{ij} may be formally solved resulting in an explicit relation, see *e.g.* Pope (1975) and Gatski & Speziale (1993).

The material derivative that includes advection by the mean flow (in the following denoted D/Dt) of a scalar field is invariant of the choice of co-ordinate system. However, the material derivative of a tensor field of higher order than zero, *e.g.* vectors and second order tensors, is *not* invariant of the choice of co-ordinate system for representing the tensor components.

It has been suggested by *e.g.* Rodi & Scheurer (1983) that the weak equilibrium assumption is better evaluated for the anisotropy tensor expressed in a streamline-based co-ordinate system. In *e.g.* circular flows where the azimuthal direction is homogeneous,

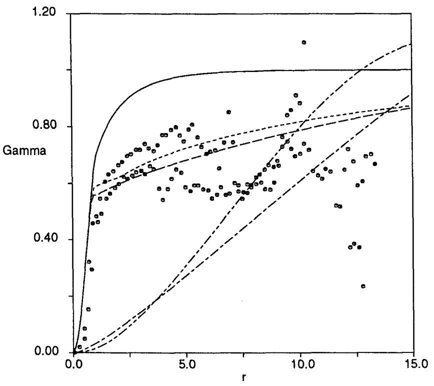

Figure 1: Computed vortex circulation $\Gamma \equiv 2\pi r V$ compared to field measurements (Campbell *et al.* 1996, 1997) with the initial profile (——). RST (– – –) compared to the corresponding EARSM with (------) and without (–·–·–) streamline correction and standard $K - \varepsilon$ model (–·–·–). Taken from Wallin & Girimaji (2000).

the weak equilibrium assumption is then exactly fulfilled. In cases with modestly curved streamlines the choice of co-ordinate system has a rather minor effect, see *e.g.* Rumsey, Gatski & Morrison (1999) for flow over an airfoil. However, in cases with strong streamline curvature this effect is dominating. In a study of a generic wing tip far-field vortex by Wallin & Girimaji (2000) it was found that the turbulent dissipation of the vortex was by far overpredicted using the standard algebraic Reynolds stress models while including the effect of the streamline curvature gave a qualitatively correct behaviour (see figure 1).

CURVATURE-CORRECTED MODEL

In this section we will formulate a streamline curvature correction of explicit algebraic Reynolds stress models (EARSM). The correction is derived from a formal evaluation of the weak equilibrium assumption in a streamline-based co-ordinate system, following ideas of Girimaji (1997) and Sjögren (1997).

General quasi-linear Reynolds stress transport models may be written in terms of a transport equation for the anisotropy tensor

$$\tau\left(\frac{\mathrm{D}a_{ij}}{\mathrm{D}t} - \mathcal{D}_{ij}^{(a)}\right) = A_0\Big[\left(A_3 + A_4\frac{\mathcal{P}}{\varepsilon}\right)a_{ij} + A_1 S_{ij} - \left(a_{ik}\Omega_{kj} - \Omega_{ik}a_{kj}\right) + A_2\left(a_{ik}S_{kj} + S_{ik}a_{kj} - \tfrac{2}{3}a_{kl}S_{lk}\delta_{ij}\right)\Big] \quad (1)$$

see Wallin & Johansson (2000). $\mathcal{D}_{ij}^{(a)}$ is the diffusion of a_{ij} and $\tau = K/\varepsilon$ is the turbulent timescale. The strain and rotation rate tensors, S_{ij} and Ω_{ij}, are normalized by τ. All linear (or quasi linear) models may be included into this form.

The weak-equilibrium assumption

Usually, in deriving algebraic Reynolds stress models the *l.h.s.* of (1) is neglected in the computational (=Cartesian) co-ordinate system. The resulting algebraic relation may be formally solved leading to an EARSM, that is an explicit relation for a_{ij}, see Wallin & Johansson (2000). The A_0 coefficient in (1) influence the EARSM only if a contribution of the *l.h.s.* of (1) is included into the EARSM.

In this section we will derive the weak-equilibrium assumption in a streamline based co-ordinate system, which is a unique base for transformation of a_{ij} independent of, and, thus, invariant of, the computational co-ordinate system. A streamline is here taken as the path of a fluid particle, though the formally correct name for this is *pathline.*

Let us define a streamline based orthogonal co-ordinate system by $\boldsymbol{e}_s \equiv (\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{s}})$. $\hat{\boldsymbol{t}}$ is in the streamline direction, $\hat{\boldsymbol{n}}$ normal to $\hat{\boldsymbol{t}}$ in the local plane of the streamline and $\hat{\boldsymbol{s}}$ is the third direction. The Cartesian (or computational) co-ordinate system $\boldsymbol{e} \equiv (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ can be transformed to $\boldsymbol{e}_s$ using the orthogonal transformation $\boldsymbol{T}$ ($\boldsymbol{T}^t\boldsymbol{T} = \mathbf{I}$)

$$\boldsymbol{e}_s = \boldsymbol{T}\boldsymbol{e} \quad (2)$$

The material derivative of the anisotropy in the streamline based co-ordinate system, $\boldsymbol{a}_s = \boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t$, transformed back to the Cartesian system, $\boldsymbol{T}^t(\cdots)\boldsymbol{T}$, may be derived

$$\boldsymbol{T}^t\frac{\mathrm{D}\,\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} = \frac{\mathrm{D}\,\boldsymbol{a}}{\mathrm{D}t} + \boldsymbol{a}\frac{\mathrm{D}\,\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} + \boldsymbol{T}^t\frac{\mathrm{D}\,\boldsymbol{T}}{\mathrm{D}t}\boldsymbol{a}. \quad (3)$$

It can be shown that for any orthogonal transformation $\boldsymbol{T}$ the following holds

$$\frac{\mathrm{D}\,\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} = -\boldsymbol{T}^t\frac{\mathrm{D}\,\boldsymbol{T}}{\mathrm{D}t} = \boldsymbol{\Omega}^{(r)} \quad (4)$$

where $\boldsymbol{\Omega}^{(r)}$ is an antisymmetric tensor. This gives

$$\frac{\mathrm{D}\,\boldsymbol{a}}{\mathrm{D}t} = \boldsymbol{T}^t \frac{\mathrm{D}\,\boldsymbol{T}\boldsymbol{a}\boldsymbol{T}^t}{\mathrm{D}t}\boldsymbol{T} - \left(\boldsymbol{a}\boldsymbol{\Omega}^{(r)} - \boldsymbol{\Omega}^{(r)}\boldsymbol{a}\right). \quad (5)$$

The best algebraic approximation may be obtained by neglecting the transformed derivative (first term at *r.h.s.* of (5)) for a specific choice of co-ordinate system. The term $(\boldsymbol{a}\boldsymbol{\Omega}^{(r)} - \boldsymbol{\Omega}^{(r)}\boldsymbol{a})$ may be fully accounted for and included into the EARSM formulation simply by replacing $\boldsymbol{\Omega}$ in (1) with

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega} - \frac{\tau}{A_0}\boldsymbol{\Omega}^{(r)}. \quad (6)$$

The formal transformation of the material derivative has earlier been presented by Girimaji (1997) and Sjögren (1997), the later with an extension for non-orthogonal co-ordinates. They extended this analysis with a formal derivation of the $\boldsymbol{\Omega}^{(r)}$ tensor from the definition of the co-ordinate system. That analysis becomes rather tedious especially in general three-dimensional flows. The final expression includes derivatives of the metrics of the curvilinear co-ordinate system which may cause numerical problems when utilized for real three-dimensional numerical computations (personal communications T. Rung, Technical University of Berlin).

In this paper we will not go further into the derivation of $\boldsymbol{\Omega}^{(r)}$, but rather try to understand the physical interpretation of $\boldsymbol{\Omega}^{(r)}$. The material derivative of $\boldsymbol{e}_s$ is expressed by using the transformation $\boldsymbol{T}$

$$\frac{\mathrm{D}\,\boldsymbol{e}_s}{\mathrm{D}t} = \frac{\mathrm{D}\,\boldsymbol{T}}{\mathrm{D}t}\boldsymbol{T}^t\boldsymbol{e}_s = -\boldsymbol{\Omega}_s^{(r)}\boldsymbol{e}_s \quad (7)$$

where $\boldsymbol{\Omega}_s^{(r)} = \boldsymbol{T}\boldsymbol{\Omega}^{(r)}\boldsymbol{T}^t$ is $\boldsymbol{\Omega}^{(r)}$ transformed to the streamline based co-ordinate system. The $\boldsymbol{\Omega}^{(r)}$ tensor is, thus, directly related to the rotation rate of the co-ordinate system following the fluid particle

$$\Omega_{ij}^{(r)} = -e_{ijk}\omega_k \quad (8)$$

where $\boldsymbol{\omega} = \omega\hat{\boldsymbol{s}} + \phi\hat{\boldsymbol{t}}$ is the co-ordinate system rotation rate vector. ω and ϕ are the rotation rates around the $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{t}}$ axes. There is no rotation around the $\hat{\boldsymbol{n}}$ axis since that is defined to be in the local plane of the streamline.

Streamline based co-ordinate system

In general three-dimensional computations it is not possible to a priori know the local streamline co-ordinate system, which must be derived from the computed flow field. The next step is, thus, to obtain a method of deriving the streamline direction. A natural choice is to use the velocity vector as the direction of the streamline. However, the velocity vector is not Galilean invariant (for a superimposed velocity), which is a necessary quality for a turbulence model.

It was proposed by Girimaji (1997) to use the acceleration vector $\dot{\boldsymbol{U}} \equiv \mathrm{D}\boldsymbol{U}/\mathrm{D}t$ as the basis for the co-ordinate system. Since the acceleration vector is Galilean invariant the resulting streamline curvature corrected model would also be Galilean invariant. Girimaji further suggested to let one of the other unit vectors be orthogonal to $\dot{\boldsymbol{U}}$ in the plane of $\dot{\boldsymbol{U}}$. In order to determine that plane in general three-dimensional flows, one needs to obtain $\ddot{\boldsymbol{U}} \equiv \mathrm{D}\dot{\boldsymbol{U}}/\mathrm{D}t$. $\dot{\boldsymbol{U}}$ and $\ddot{\boldsymbol{U}}$ are then used to form the local co-ordinate system and in the further, quite complex, formal derivation of the $\boldsymbol{\Omega}^{(r)}$ tensor one additional derivation of the co-ordinate system base is needed.

Is it possible to approximate the co-ordinate system rotation rate, $\boldsymbol{\omega}$, directly from the acceleration vector and the rate of change of that? Let us investigate the following approximation

$$\boldsymbol{\omega}^{(\mathrm{approx})} = \frac{\dot{\boldsymbol{U}} \times \ddot{\boldsymbol{U}}}{\dot{\boldsymbol{U}}^2}. \quad (9)$$

The velocity along the streamline is $\boldsymbol{U} = V\hat{\boldsymbol{t}}$. If the magnitude of the velocity along the streamline, V, is constant, the acceleration vector becomes $\dot{\boldsymbol{U}} = \omega V\hat{\boldsymbol{n}}$ and is normal to the streamline. The material derivative of the acceleration is then derived by using (7) and (8) to

$$\ddot{\boldsymbol{U}} = \frac{\mathrm{D}\,\dot{\boldsymbol{U}}}{\mathrm{D}t} = \dot{\omega}V\hat{\boldsymbol{n}} - \omega^2 V\hat{\boldsymbol{t}} + \omega\phi V\hat{\boldsymbol{s}} \quad (10)$$

and the approximation $\boldsymbol{\omega}^{(\text{approx})}$ in (9) becomes

$$\boldsymbol{\omega}^{(\text{approx})} = \omega\hat{\boldsymbol{s}} + \phi\hat{\boldsymbol{t}} = \boldsymbol{\omega} \qquad (11)$$

which is identical to the expression for the total rotation rate vector from the formal transformation when $\dot{V} = 0$. If the magnitude of the velocity along the streamline is not constant, additional terms appear in the approximation (see below).

The limit of vanishing curvature

One important property of a streamline curvature correction is that it should vanish when the streamline approaches a straight line. Let us consider an almost straight streamline where the curvature ω is small. The acceleration then becomes $\dot{\boldsymbol{U}} = \dot{V}\hat{\boldsymbol{t}} + \omega V\hat{\boldsymbol{n}}$ and the rate of change of that $\ddot{\boldsymbol{U}} = \ddot{V}\hat{\boldsymbol{t}} + (2\omega\dot{V} + \dot{\omega}V)\hat{\boldsymbol{n}} + \omega\phi V\hat{\boldsymbol{s}}$ (plus higher order terms in ω). The approximative ω by use of (9) then becomes

$$\boldsymbol{\omega}^{(\text{approx})} = \omega\left[1 + \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{V}{\dot{V}}\right) + \frac{\dot{\omega}}{\omega}\frac{V}{\dot{V}}\right]\hat{\boldsymbol{s}} + \mathcal{O}(\omega^2) \qquad (12)$$

Hence, $\boldsymbol{\omega}^{(\text{approx})} = \mathcal{O}(\omega)$ for most cases, and thus approaches zero as the streamline curvature vanishes. Note that if $\dot{V} = 0$ we retrieve the exact result for $\boldsymbol{\omega}^{(\text{approx})}$ (see discussion in previous section).

However, if $\dot{\boldsymbol{U}}$ approaches zero the approximation becomes ill posed. Fortunately, in these cases also the curvature approaches zero and by limiting the denominator in (9) to some small number C_r (times ε^2/K) a well-posed approximation results

$$\boldsymbol{\omega}^{(\text{approx})} = \frac{\dot{\boldsymbol{U}} \times \ddot{\boldsymbol{U}}}{\max\left(\dot{\boldsymbol{U}}^2, C_r\varepsilon^2/K\right)}. \qquad (13)$$

GENERIC TEST CASES

Rotating homogeneous shear flow

Rotating homogeneous shear flow may be used as an illustration of the effect of including the streamline curvature correction. In this specific case it is obvious to transform the

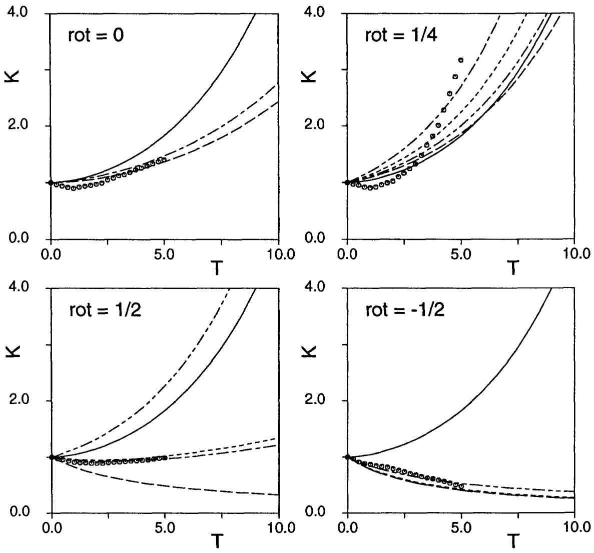

Figure 2: Computed evolution of the turbulent kinetic energy K in rotating homogeneous shear flow compared to large eddy simulation (∘), Bardina *et al.* (1983). Curvature corrected WJ (– – –), CC-WJ (------) and L-SSG (–·–) EARSMs compared to non-corrected CC-WJ EARSM (–·–·) and eddy-viscosity model (——). $\tau dU/dy = 3.4$ initially.

anisotropy tensor to the rotating co-ordinate system, but exactly the same effect is obtained by applying the correction proposed in this paper. For this case, the weak equilibrium assumption is exactly fulfilled.

Four different cases were computed where the rotation rate $Ro \equiv \omega/(\mathrm{d}U/\mathrm{d}y)$ is 0, 1/4, 1/2 and $-1/2$, see figure 2. It is obvious that the eddy-viscosity model cannot distinguish between the different rotation rates.

These cases were computed with the curvature corrected algebraic Reynolds stress model resulting from (1) with model coefficients given in table 1. The model based on the linearized SSG (L-SSG) gives reasonable growth rates for most rotation numbers, though underestimating the most energetic case ($Ro = 1/4$). The Wallin & Johansson (2000) model underestimates both cases with positive rotation. However, by increasing the A_0 coefficient to the same value as for L-SSG the CC-WJ model

	A_0	A_1	A_2	A_3	A_4
L-SSG	−0.80	1.22	0.47	0.88	2.37
WJ	−0.44	1.20	0	1.80	2.25
CC-WJ	−0.80	1.20	0	1.80	2.25

Table 1: The values of the A-coefficients for different quasi-linear pressure-strain models

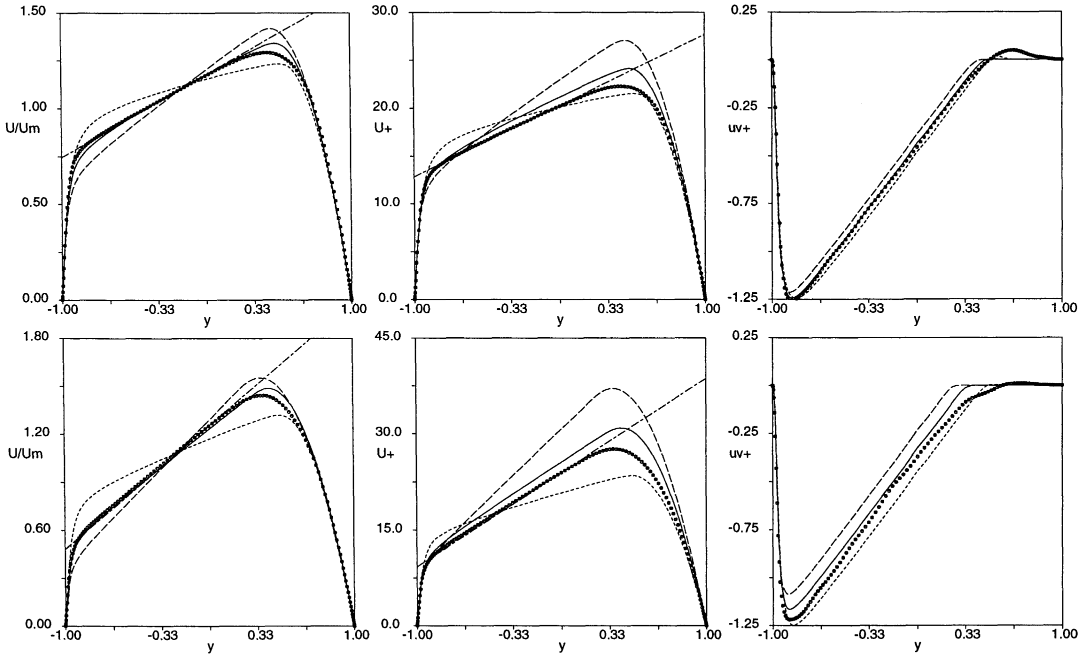

Figure 3: Computed rotating channel flow for Ro = 0.43 (top) and Ro = 0.77 (bottom) compared to DNS of Alvelius & Johansson (2000). Curvature corrected WJ (– – –) and CC-WJ (——) EARSM compared to non-corrected CC-WJ EARSM (------). $U' = 2\omega$ (–·–·–) is also shown.

gives predictions close to the L-SSG. The same was observed by Wallin & Girimaji (2000) for the vortex.

Switching off the curvature correction for the CC-WJ model degenerates the predicted growth rate for the $Ro = 1/4$ case, and for the $Ro = 1/2$ case the growth rate is severely overpredicted. From this, it is clear that the streamline curvature correction is important.

Fully developed rotating channel

Fully developed rotating channel is considered. The channel coordinate system is $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ which is rotating with the rate ω in the $\hat{\boldsymbol{z}}$ direction. Also in this case it is obvious to transform a_{ij} to the rotational frame, and both the exact transformation and the proposed approximation exactly fulfills the weak equilibrium assumption.

	DNS		CC-WJ	
Ro	0.43	0.77	0.43	0.77
Re_τ^s	129.7	133.0	129.3	137.6
Re_τ^u	218.3	217.2	218.6	213.9
Re_m	3094	3446	3237	3741

Table 2: Rotating channel flow. DNS specification (Alvelius & Johansson 2000) and computational results using the curvature corrected CC-WJ EARSM.

Direct numerical simulations of a fully developed rotating channel at different Reynolds and rotational numbers were made by Alvelius & Johansson (2000). The two most rapidly rotating cases for $Re_\tau \equiv u_\tau \delta/\nu = 180$ are computed here. δ is the half channel width and the average wall friction velocity is defined as $2u_\tau^2 = u_\tau^{s\,2} + u_\tau^{u\,2}$ where u_τ^s and u_τ^u are the stable and unstable side friction velocities, respectively. The rotation number $Ro \equiv 2\omega\delta/U_m$ and the bulk Reynolds number $Re_m \equiv U_m\delta/\nu$ are defined in table 2 where ω is the rotation rate of the system and U_m is the bulk velocity.

The Wallin & Johansson (2000) EARSM based on the Wilcox (1988) $K - \omega$ model is computed with the proposed curvature correction. The curvature corrected CC-WJ EARSM agrees well with DNS data, though a slightly overpredicted Re_m is seen in the U^+ plots in figure 3 and in table 2. Using the curvature corrected original WJ EARSM the effect of rotation is slightly overestimated while the non-corrected CC-WJ EARSM underpredicts the rotation effects. Thus, the effect of curvature correction and the choice of A_0 is also for this case important. In table 2 it is seen that the skin friction differences between the stable and unstable sides are well captured by the curvature corrected CC-WJ model.

The shear stress plots in figure 3 show that all models reasonable well captures the laminarization on the stable side. However, the DNS data show a small level of positive shear stress for the $Ro = 0.43$ case where all models gives almost zero $\overline{uv}$.

CONCLUDING REMARKS

The proposed curvature correction is a straight forward extension of existing EARSMs and it is important to note that the original EARSM is retrieved in cases without streamline curvature. The proposed correction is explicitly expressed in terms of the acceleration vector and its material derivative. Thus, second derivatives of the velocity field are needed. That should not cause any major numerical problems, since that is already needed for the diffusion term in the momentum equation. However, the matter of computational stability is an important aspect that will be addressed in further studies.

The acceleration vector and its material derivative fulfill Galilean invariance, that is independence of solid-body motion of the frame of reference, and, thus, also the proposed correction is invariant. However, any incompressible flow field should also be independent of a superimposed solid-body constant acceleration, according to Spalart & Speziale (1999), except for a modified pressure field. The proposed modification must thus be used with caution in accelerated frames of reference. Extensions of EARSMs for including approximations of the usually neglected transport terms could never be expected to be completely general, but could anyway be motivated by improved model performance in a reasonably wide class of flows.

An alternative approach was suggested by Gatski & Jongen (2000) where the transformation is derived from the eigenvectors of the mean strain rate tensor for two-dimensional mean flows. However, it is not completely clear how to extend that to general three-dimensional mean flows.

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