DIRECT STUDY OF THE CONSTITUTIVE EQUATION FOR TURBULENT NEAR-WALL FLOWS USING DNS DATA

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ABSTRACT
Eddy-viscosity turbulence models rest on a constitutive equation providing the Reynolds stress tensor from the mean velocity gradient tensor, together with transport equations for turbulent quantities such as k or ε. The classical constitutive equation is a linear stress-strain relation, leading to linear k-ε type models, which have clear limitations to predict complex flows. Nonlinear generalizations have been proposed, but have not often been directly tested. Here we consider a quadratic nonlinear constitutive equation; in this framework we focus on the determination of its coefficients. For this we use DNS data of simple shear flows and directly show that nonlinear constitutive equations are necessary, and study their coefficients using invariants.

INTRODUCTION: BOUSSINESQ'S HYPOTHESIS
Reynolds (or Favre) averaging of Navier-Stokes equation involves the Reynolds stress tensor $\mathbf{T} = -\rho < u_i u_j >$, where ρ is the density and bold notations are used for tensors, (u_i) is the fluctuating velocity and (U_i) the mean velocity. To achieve closure the stress tensor must be expressed from mean velocity quantities.

Let us denote $k = -\frac{1}{2\rho}\{\mathbf{T}\}$ the turbulent kinetic energy, where $\{\mathbf{X}\}$ represents the trace of the matrix (or tensor) $\mathbf{X}$, and the traceless stress tensor: $\mathbf{R} = \frac{1}{\rho}\mathbf{T} + \frac{2}{3}k\mathbf{I}$. We introduce also the velocity gradient tensor $\mathbf{A} = \partial U_i / \partial x_j$ and its symmetric traceless part, the mean strain tensor:

$$\mathbf{S} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{1}{3}\{\mathbf{A}\}\mathbf{I} \qquad (1)$$

In case of incompressibility, $\{\mathbf{A}\} = 0$. Boussinesq's hypothesis (Boussinesq, 1877) corresponds then to a closure hypothesis with the following linear constitutive equation:

$$\mathbf{R} = \nu_T \mathbf{S} \qquad (2)$$

where ν_T is a scalar called the eddy viscosity. This linear relation between stress and strain is an important hypothesis, coming from an analogy with the constitutive equation for Newtonian fluids, involving the viscous stress tensor and the viscosity ν: $\mathbf{R}_\nu = \nu\mathbf{S}$.

In Equation (2), the eddy-viscosity is written for classical k-ε model (Launder and Spalding, 1974) using two independent turbulent quantities such as k and the dissipation ε: $\nu_T = C_\mu k^2/\varepsilon$, where C_μ is a non-dimensional quantity (in some recent models it is no more constant). Closure is achieved with transport equations for k and ε. Here we do not consider transport equations and focus on the constitutive equation.

The k-ε two-equations turbulence models are widely used for turbulence modeling in engineering, despite their clear shortcomings for complex flow prediction (see for reviews Wilcox, 1998; Piquet, 1999; Pope, 2000). Generalizations have involved modifications of transport equations, and in parallel the replacement of the linear relation (2) by nonlinear constitutive equations. We present the latter below before discussing them directly using numerical and experimental data.

NONLINEAR CONSTITUTIVE EQUATIONS

Anisotropy of the Reynolds stress tensor
One of the main deficiencies of linear models is their inability to produce an anisotropic Reynolds stress tensor for simple shear flows. As a simple example, when only one component of the velocity gradient tensor is non-zero (dU(y)/dy=a) and the shear stress $\tau = -< uv >$, we have:

$$\mathbf{R} = \begin{pmatrix} \frac{2}{3}k - < u^2 > & \tau & 0 \\ \tau & \frac{2}{3}k - < v^2 > & 0 \\ 0 & 0 & \frac{2}{3}k - < w^2 > \end{pmatrix} \qquad (3)$$

and

$$\mathbf{S} = \frac{a}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

showing that Equation (2) can be verified only when the diagonal of **R** vanishes, corresponding to isotropic normal stresses (see e.g. Nisizima and Yoshizawa, 1986; Speziale, 1987). Whenever normal stresses are anisotropic for these simple shear situations, Boussinesq's hypothesis does not apply. Since most experimental and DNS databases present anisotropic normal stresses, it can be seen that even for simple shear flows a nonlinear constitutive equation is needed.

Nonlinear constitutive equation: general 3D case. Since Pope (1975), it is recognized that invariant theory (and especially results obtained in the fifties by Rivlin and Spencer, see Spencer, 1971) can be used in the framework of turbulence modeling to represent the stress tensor as a development into a tensor basis composed of no more than 10 basis tensors. This provides the following general development:

$$\mathbf{R} = \sum_{i=1}^{10} a_i \mathbf{T}_i \tag{5}$$

where a_i are scalar functions and the tensors of the basis are given by:

$$\begin{array}{ll} \mathbf{T}_1 = \mathbf{S} & \mathbf{T}_6 = \mathbf{W}^2\mathbf{S} + \mathbf{S}\mathbf{W}^2 - \frac{2\eta_4}{3}\mathbf{I} \\ \mathbf{T}_2 = \mathbf{SW} - \mathbf{WS} & \mathbf{T}_7 = \mathbf{WSW}^2 - \mathbf{W}^2\mathbf{SW} \\ \mathbf{T}_3 = \mathbf{S}^2 - \frac{\eta_1}{3}\mathbf{I} & \mathbf{T}_8 = \mathbf{SWS}^2 - \mathbf{S}^2\mathbf{WS} \\ \mathbf{T}_4 = \mathbf{W}^2 - \frac{\eta_2}{3}\mathbf{I} & \mathbf{T}_9 = \mathbf{W}^2\mathbf{S}^2 + \mathbf{S}^2\mathbf{W}^2 - \frac{2\eta_5}{3}\mathbf{I} \\ \mathbf{T}_5 = \mathbf{WS}^2 - \mathbf{S}^2\mathbf{W} & \mathbf{T}_{10} = \mathbf{WS}^2\mathbf{W}^2 - \mathbf{W}^2\mathbf{S}^2\mathbf{W} \end{array} \tag{6}$$

where **W**=**A**-**S** is the skew symmetric part of the velocity gradient tensor. These tensors are all symmetric and traceless, and are all supposed to be mutually independent. The scalars introduced are invariants of the flow. All the invariants involving products of strain and vorticity tensors are the following:

$$\begin{array}{ll} \eta_1 = \{\mathbf{S}^2\} & \eta_4 = \{\mathbf{W}^2\mathbf{S}\} \\ \eta_2 = \{\mathbf{W}^2\} & \eta_5 = \{\mathbf{W}^2\mathbf{S}^2\} \\ \eta_3 = \{\mathbf{S}^3\} & \end{array} \tag{7}$$

Other invariants are also needed, involving the stress tensor:

$$\begin{array}{ll} \mu_1 = \{\mathbf{R}^2\} & \mu_4 = \{\mathbf{RS}^2\} \\ \mu_2 = \{\mathbf{RS}\} & \mu_5 = \{\mathbf{RW}^2\} \\ \mu_3 = \{\mathbf{RSW}\} & \mu_6 = \{\mathbf{RWS}^2\} \end{array} \tag{8}$$

The main points characterizing the nonlinear models are the choice of a tensor basis among the 10 terms given in Equation (6) and an expression for the coefficients a_i in Equation (5) as function of k, ε and the invariants η_i. The linear constitutive equation corresponds to retaining only T_1 as tensor basis. Quadratic models correspond to the choice T_1-T_4. It has been argued that the term T_4 should not appear in the tensor basis since it leads to physically inconsistent results in rotating flows (Speziale, 1998); in this case quadratic models comprise only 3 terms. Cubic models include also tensors T_5 and T_6, corresponding to a 5-terms basis. Quartic models add three more terms (T_7- T_9) and quintic models one more term (T_{10}); these terms have not yet been added in practical model implementation, and up to now only quadratic and cubic nonlinear models have been implemented and tested.

The expressions giving the coefficients in front of the tensor basis are sometimes completely *ad hoc*, and in other occasions, extrapolated from fits performed in very special test cases. These scalar coefficients are then clearly the weak point of non-linear eddy-viscosity models, since it is of little use to have the correct tensor basis if the magnitude of the coefficients is not correctly modeled. Below we apply and adapt a projection method in order to better estimate the coefficients from the invariants.

Quadratic constitutive equation and projection onto the tensor basis. A new approach has been initiated by Jongen and Gatski (1998) in the framework of explicit algebraic stress models. This assumes an equilibrium or near-equilibrium condition (production = dissipation), which is not valid for inhomogeneous or complex flows in general. Therefore, we do not discuss here explicit algebraic stress models, but the projection procedure proposed can be applied also to nonlinear eddy-viscosity models. Better than guessing a possible form for the coefficients, Jongen and Gatski proposed a mathematical procedure to obtain an exact form for the coefficients. It consists in projecting, through a scalar product for tensors or matrices, the constitutive equation onto the tensor basis. The scalar product can be written:

$$[\mathbf{A},\mathbf{B}] = \{\mathbf{A}^t\mathbf{B}\} = A_{ij}B_{ij} \tag{9}$$

with the associated norm:

$$\|\mathbf{A}\| = [\mathbf{A},\mathbf{A}]^{1/2} = \left(A_{ij}A_{ij}\right)^{1/2} \tag{10}$$

The projection method is the following: apply successively to Equation (5) a scalar product with each term of the tensor basis. This then transforms this tensor equation into a system of scalar equations that can be solved. The left-hand side of the scalar equation involves an invariant $\{\mathbf{RT}_i\}$ which can be expressed using the invariants μ_i. All the coefficients obtained are written using invariants given in Equations (7) and (8). When inverting the linear system, the coefficients are obtained as rational function of the invariants.

In the following we consider a quadratic model with 3 terms basis. Let us rewrite the quadratic constitutive equation in the following way:

$$\mathbf{R} = \nu_T \mathbf{S} - \beta(\mathbf{SW} - \mathbf{WS}) - \gamma\left(\mathbf{S}^2 - \frac{\eta_1}{3}\mathbf{I}\right) \quad (11)$$

with the coefficients given by the system:

$$\begin{pmatrix} \mu_2 \\ 2\mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} \eta_1 & 0 & \eta_3 \\ 0 & \eta_1\eta_2 - 6\eta_5 & 0 \\ \eta_3 & 0 & \frac{1}{6}\eta_1^2 \end{pmatrix} \begin{pmatrix} \nu_T \\ -\beta \\ -\gamma \end{pmatrix} \quad (12)$$

having as solution:

$$\begin{cases} \nu_T = \dfrac{\eta_1^2\mu_2 - 6\eta_3\mu_4}{\eta_1^3 - 6\eta_3^2} \\ \beta = \dfrac{-2\mu_3}{\eta_1\eta_2 - 6\eta_5} \\ \gamma = \dfrac{6(\mu_2\eta_3 - \eta_1\mu_4)}{\eta_1^3 - 6\eta_3^2} \end{cases} \quad (13)$$

For purely 2D flows (for which **S** has one vanishing eigenvalue) some invariants given above vanish or are no more independent of others: $\eta_3 = 0$, $\eta_4 = 0$ and $\eta_5 = \eta_1\eta_2 / 2$, as shown by Jongen and Gatski (1998). Then the 3-terms development in Equation (11) is complete, the matrix in Equation (12) becomes diagonal, and the coefficients in Equation (13) have a simpler form. This results in the following simple expression, which is a general identity for 2D flows:

$$\mathbf{R} = \frac{\mu_2}{\eta_1}\mathbf{S} - \frac{\mu_3}{\eta_1\eta_2}\mathbf{T}_2 + 6\frac{\mu_4}{\eta_1^2}\mathbf{T}_3 \quad (14)$$

These expressions provide the exact coefficient functions, and can be used to provide a best approximation of these functions. This is discussed below.

Comments on nonlinear constitutive equations. First of all, the choice of the tensors is certainly as important as the choice of a functional dependence for the coefficients. It appears that most nonlinear models have so far chosen coefficients in a quite *ad hoc* manner, since:

- The first quadratic models have simply chosen constant coefficients (Yoshizawa, 1984; Speziale, 1987; Myong and Kasagi, 1990; Rubinstein and Baron, 1990).
- Other nonlinear models (e.g. Shih and Lumley, 1993; Gatski and Speziale, 1993; Craft et al., 1996; Apsley and Leschziner, 1998) have proposed variable coefficients, depending on invariants. But the functions chosen are often *ad hoc*, since they are introduced only for they limit behaviour in some special situations. The coefficients have then been calibrated in special situations (such as channel flow) assuming that they are the same for other more complex flows.

There is clearly too much arbitrariness in these choices, and it seems desirable to have a more general and rigorous method to determine the coefficients. The relations given by Equation (13) are exact but cannot be used as they are in eddy-viscosity turbulence models, since there is a dependence on the stress tensor through the invariants μ_i.

On the other hand, Equation (13) can be used to obtain the coefficients from experimental or DNS data (see some first application in Schmitt and Hirsch, 2000; 2001). General functions depending on the geometry, k, ε and the invariants η_i, can be extracted from the data, in a way which can then be implemented into eddy viscosity turbulence models. We apply below this procedure to DNS data corresponding to simple shear flows near the wall.

APPLICATION TO DNS DATA OF SIMPLE SHEAR FLOWS

Superposition of turbulent profiles for different DNS databases. As a direct application of the previous procedure, we use DNS data of simple shear flows possessing only one non-zero velocity gradient (see also Schmitt and Hirsch, 2001). The first data set (denoted CO in the following) corresponds to a turbulent plane Couette flow at a Reynolds number Re=1300, with a friction Reynolds number Re_τ=82 (Bech et al, 1995). The second data set (denoted CF87 in the following) corresponds to a turbulent channel flow at a Reynolds number Re=3250 corresponding to a friction Reynolds number of Re_τ=180 (Kim, Moin and Moser, 1987), and the third (denoted CF99) is the same flow at larger Reynolds number: $Re \approx 10^4$ corresponding to a friction Reynolds number of Re_τ=590 (Moser, Kim and Mansour, 1999). The last data set (denoted BL in the following) is a turbulent boundary layer on a flat plate, with zero pressure gradient, with a Reynolds number of $Re \approx 2.10^5$, or based on the momentum thickness θ, Re_θ=1410 (Spalart, 1988). The test cases chosen correspond to a panel of Reynolds numbers going from 10^3 to 10^5. The friction Reynolds numbers above are defined using the friction velocity $u_\tau = \sqrt{\tau_w / \rho}$ where τ_w is the wall shear stress. In the following, all the quantities are non-dimensionalized, using u_τ or u_τ^2; the distance to the wall is expressed as usual in wall units (also a local Reynolds number) $y^+ = yu_\tau / \nu$.

Figure 1 shows the mean velocity profiles $U(y^+)$ for the inner layer ($y^+ < 500$) for the 4 data sets, showing the log-law and the viscous regions. Figure

2 represents the normal stresses for the 4 data sets, showing that the anisotropy of the stresses is pronounced and indicating indeed that, as discussed above, a nonlinear constitutive equation is needed. Figure 3 shows the shear stress τ. We first observe that, despite the different flow configurations and different Reynolds numbers, the profiles in Figures 1-3 superpose rather well for small values of y^+: this is especially true for the mean velocity profile; the superposition is also quite good for the stresses. The behavior shown in Figures 2 and 3 for small y^+ is consistent with classical results obtained from Taylor expansion: $<u^2>$ and $<w^2>$ are proportional to the square of y^+, while $<v^2>$ and τ are proportional to its third power (for $y^+<10$).

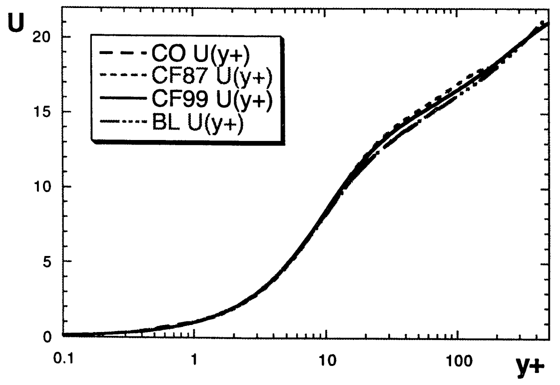

Figure 1 : Mean velocity profiles $U(y^+)$.

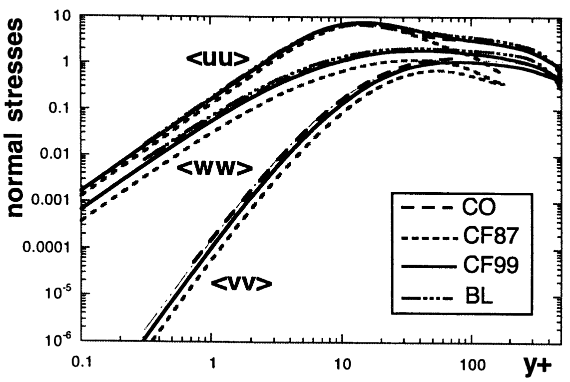

Figure 2 : Normal stresses in log-log plot.

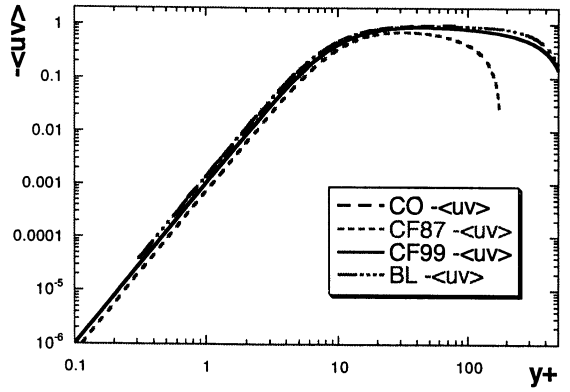

Figure 3 : The shear stress τ=-<uv>.

Estimation of the coefficients in a quadratic constitutive equation. As a direct test of Boussinesq's hypothesis, we first represent here the normalized scalar product of **R** and **S** tensors (it can be also interpreted as the cosine of an "angle" between **R** and **S**, see Schmitt and Hirsch, 2000; 2001):

$$\rho_{RS} = \frac{\{\mathbf{RS}\}}{\|\mathbf{R}\|\|\mathbf{S}\|} = \frac{\mu_2}{\sqrt{\eta_1 \mu_1}} \tag{15}$$

where μ_2 is the kinetic energy production. The different invariants (defined in Equations (7) and (8)) can be estimated from the turbulence statistics. We provide here some of them, which will be useful in the following. These are:

$$\eta_1 = \frac{a^2}{2} \; ; \; \eta_2 = -\frac{a^2}{2} \; ; \; \mu_2 = a\tau$$

$$\mu_3 = -\frac{a^2}{4}\left(<u^2> - <v^2>\right) \tag{16}$$

$$\mu_4 = -\frac{a^2}{4}\left(\frac{2}{3}k - <w^2>\right)$$

where a=dU/dy and k is the kinetic energy. Equations (14) and (16) then give:

$$\nu_T = \frac{\mu_2}{\eta_1} = \frac{2\tau}{a}$$

$$\beta = \frac{\mu_3}{\eta_1 \eta_2} = \frac{1}{a^2}\left(<u^2> - <v^2>\right) \tag{17}$$

$$\gamma = \frac{-6\mu_4}{\eta_1^2} = \frac{6}{a^2}\left(\frac{2}{3}k - <w^2>\right)$$

Equations (15) and (16) provide ρ_{RS}, which is represented in Figure 4; in the estimation of the Reynolds stress, we included the normalized viscous stress $\mathbf{R}_v / \nu = \mathbf{S}$, so that for very small values of y^+, the viscous limit is reached. Since $\rho_{RS}=1$ for Boussinesq's hypothesis, the smaller ρ_{RS}, the more nonlinear terms are important. This figure directly shows that for $y^+>2$, nonlinear terms are needed. This parameter is especially small for $3<y^+<70$, showing that for these distances to the wall (corresponding roughly to the buffer layer) the linear term is of little relative weight. There is a minimum at $y^+ = 10.5 \pm 0.5$.

Figures 5-7 show the 3 coefficients in Equation (17) in log-log plots. The rather good superposition obtained for the 4 data sets indicates that these coefficients may have some universal form for near-wall flows and for various Reynolds numbers, at least in the inner layer considered here. Fits of these experimental curves may be easily performed, providing a general formulation for the nonlinear constitutive equation for near-wall turbulence. We have performed power-law fits for small values of y^+: as given by Taylor-expansion (see Notter and Sleicher, 1971; Patel, Rodi and Scheuerer, 1985), a behaviour of $\nu_T / \nu \propto (y^+)^3$ is detected for $y^+<15$. The coefficient b varies as $\beta \propto (y^+)^2$ for $y^+<15$, as expected from the fact that a=1 close to the wall, with the Taylor expansions $< u^2 > \propto (y^+)^2$ and $< v^2 > \propto (y^+)^3$ (see e.g. Pope, 2000). On the other hand, we find for γ a development as $\gamma \propto (y^+)^{2.5}$ for the whole viscous wall region ($y^+<50$), different from the $(y^+)^2$ law expected from Taylor expansion (this law is recovered for 2 datasets for $y^+<1$).

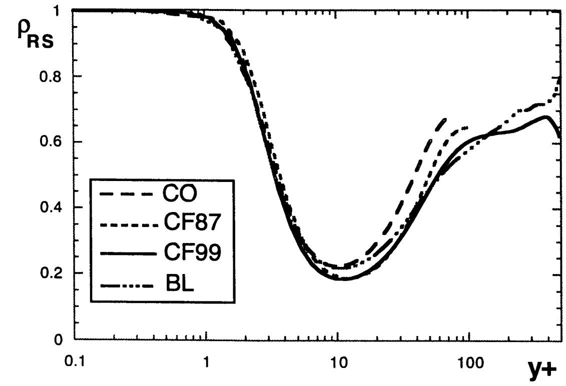

Figure 4 : The ratio ρ_{RS}, measure of the nonlinear terms.

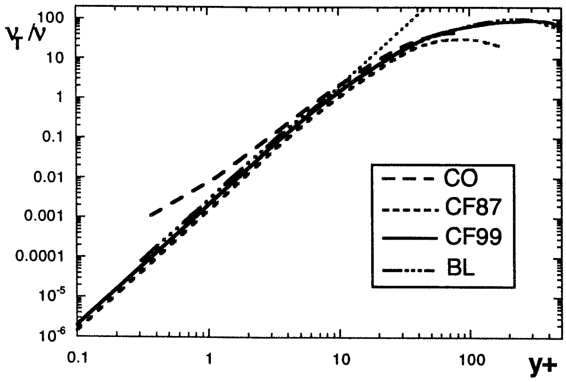

Figure 5 : The normalized eddy-viscosity ν_T / ν in log-log plot. The equation of the straight line is $0.002(y+)^3$.

A mixing-length formulation with no viscous damping function. Let us finally add a comment about mixing-length formulation for the eddy-viscosity. With our notation, this is usually written as:

$$\frac{\nu_T}{\nu} = (\ell_m)^2 a \tag{18}$$

Since a=1 in the viscous sublayer ($y^+<5$), the cubic development for the eddy-viscosity corresponds simply to a development for the mixing-length close to the wall:

$$\ell_m = \ell_0 \left(\frac{y^+}{y_0} \right)^{3/2} \tag{19}$$

where ℓ_0 and y_0 are constants. With this formulation, no Van Driest damping function is needed close to the wall to recover the correct expansion for the eddy-viscosity. Using this expression for the mixing-length, the velocity profile given by (see e.g. Pope, 2000):

$$U^+(y^+) = \int_0^{y^+} \frac{2dr}{1 + \sqrt{1 + 4(\ell_m(r))^2}} \tag{20}$$

has a development for small y^+ as

$$U^+(y^+) = y^+ - A(y^+)^4 + .. \tag{21}$$

as expected (see Pope, 2000, p. 288).

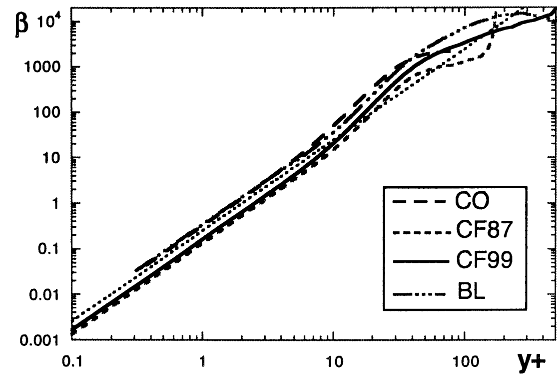

Figure 6 : The coefficient β in log-log plot. The equation of the straight line is $0.25(y+)^2$.

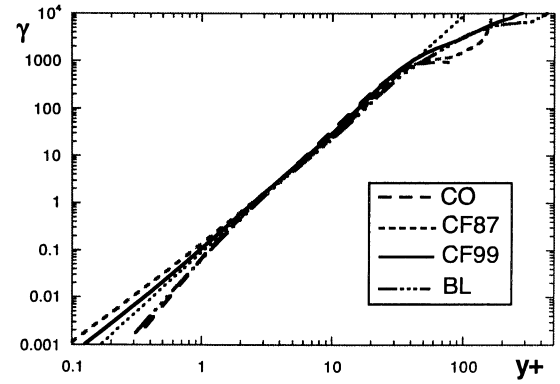

Figure 7 : The coefficient γ in log-log plot. The equation of the straight line is $0.09(y+)^{2.5}$.

CONCLUSIONS

We directly considered the constitutive equation for 4 different DNS databases corresponding to 3 different near-wall flows (boundary-layer, channel flow, Couette flow), at Reynolds numbers ranging from 10^3 to 10^5. We first showed that, except very close to the wall ($y^+<2$), the linear constitutive equation is not adequate, contrary to what is often implicitly assumed. We have further verified that the coefficients in a quadratic development are similar for the 4 test cases. This corresponds to a new type of experimental wall functions, which are given directly for a nonlinear constitutive equation. This may directly be implemented into turbulence models.

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