A CONJECTURE: TURBULENCE SUBJECTED TO CYCLIC STRAIN ALWAYS DECAYS

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ABSTRACT
Analytical solutions of a second moment closure for homogeneous turbulence subjected to periodic compression-dilatation strains show that both the characteristic turbulence frequency and turbulent kinetic energy eventually decay, irrespective of the initial state. The eddy-viscosity models on the other hand give erroneous results because they overlook the phase lag between periodic strain rate and stresses in the exact production term in the second moment closure. The author's opinion that, turbulence submitted to cyclic strains should always finally decay, remains a conjecture however, since some (justified) simplifications were necessary to produce the analytical solutions.

INTRODUCTION
For a sudden strong strain, the rapid distortion theory (RDT) predicts an immediate response of the turbulence stress field, with the components of the stress tensor aligned with the imposed strain rate tensor (see Hunt & Carruthers 1990, Cambon & Scott 1999, for recent reviews of RDT which was introduced by Batchelor & Proudman 1954). Turbulence models, irrespective of the modeling level, yield similar results. While these solutions reflect reasonably well the reality for a short time after the strain is imposed, later behaviour of the stress field departs from the RDT. Spectral and spatial transport, pressure scrambling and viscous dissipation become all important in the dynamics of the stress field, resulting in a response phase shift, which differs from one stress component to another. In the case of a cyclic variation of the imposed strain, the variation of the stress field will exhibit a hysteresis which depends on the local flow conditions and will vary over the flow (Hanjalić et al 1995).

The most relevant engineering application of a cyclic strain field is in the cylinders of internal combustion engines. Current industrial CFD (Computational Fluid Dynamics) codes for predicting ICE flow use invariably the standard $k-\varepsilon$ eddy viscosity (diffusivity) turbulence models (EVM) to close the averaged momentum and energy equations. In this paper we demonstrate that such a practice leads inevitably to erroneous results because of inadequate modelling of the generation of turbulence by the irrotational strain. While this is only one among several sources of turbulence (shearing strain, swirl and tumble, wall effects), its inadequate treatment can not but result in faulty solutions. While the conventional differential second-moment closures (SMC) still suffer from several deficiencies (the use of a single characteristic turbulence scale, uncertainties in modelling the pressure-scrambling effect), the exact treatment of the stress production offers decisive advantages, particularly in flows dominated by irrotational (normal) strains.

In this paper we confine our attention to the analysis of the effect of the periodic compression and expansion on the turbulence field by considering a shear-free homogeneous turbulence, away from a solid wall. Although far from a real engine, this idealized flow situation allows both the analytical and simple numerical solutions of the problem, illustrating the importance of the exact treatment of the turbulence stress generation mechanism.

HOMOGENEOUS IRROTATIONAL CYCLIC STRAIN: CONJECTURE
We consider first the evolution of the turbulence kinetic energy and the characteristic time scale in a general case of homogeneous turbulence subjected to a cyclic compression and expansion, excluding secondary motions and wall effects. We thus consider a control volume in the core of a very large chamber. The problem is then homogeneous, with an imposed cyclic rate of strain $S_{ij} = (S_{11}, 0, 0) = dU/dx$, where $S_{ij} = 1/2(\partial U_i/\partial x_j + \partial U_j/\partial x_i)$ is the mean rate of strain tensor, U is the local mean flow velocity and x denotes the direction of piston movement. Since no external length-scale is given, all relevant parameters scale with time. The external parameter is the piston motion period T. The time t is non-dimensionalized by the period, T , here chosen for convenience to be $T = 1$. The turbulent inverse time-scale is defined by $f = \varepsilon/k$ with an initial value of $f_0 = \varepsilon_0/k_0$. Here $k = 1/2\overline{u_i u_i}$ is the turbulence kinetic energy, u_i is the velocity fluctuation around a mean, and ε is the dissipation rate of k. In this paper we use ε as a scale-providing variable obtainable from the solution of the model evolution equation, as implied by the standard $k-\varepsilon$ and second-moment closures (SMC). The turbulence decay is characterized by f tending toward zero (turbulence time-scales going to infinity).

When the non-dimensional strain parameter $\eta = Sk/\varepsilon = S/f$ is large ($S = \sqrt{1/2 S_{ij} S_{ji}}$), this means that the eddy turn-over time is very short compared to the time scale of the imposed strain rate. The rapid distortion limit can then be applied, i.e. the linear or "production" terms involving S are predominant compared to the non-linear or "slow" terms characterizing return to isotropy and dissipation effects.

THE STRAIN RATE

In an internal combustion engine with the compression ratio r, the rate of cyclic strain obtained from the piston velocity V_1 is:

$$S_{11}(t) = \frac{dV_1}{dx_1} = \frac{\omega\, cos(\omega t)}{sin(\omega t) + \left(1 + \frac{1}{r}\right)} \quad (1)$$

where $T = 2\pi/\omega$ is the cycle period. We consider a typical case where, for example, the piston stroke covers $1 - 1/r = 4/5$ of the cylinder length, so that the strain rate $S(t)$ and its primitive, the cumulative strain, $I(t) = \int S\, dt$, are, respectively:

$$S(t) = \frac{\omega \cos(\omega t)}{\sin(\omega t) + 5/4} \quad (2)$$

$$I(t) = \ln(\sin(\omega t) + \frac{5}{4}) \quad (3)$$

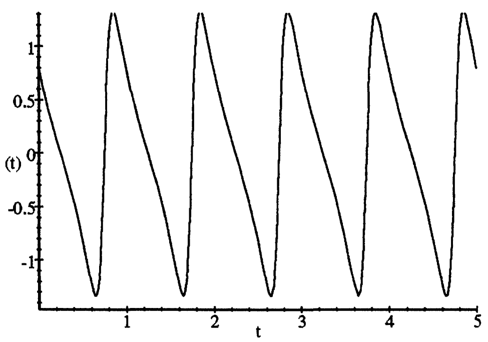

Figure 1. Strain rate $S(t)$ imposed.

KINETIC ENERGY AND TIME-SCALE SOLUTIONS

The equations for energy and dissipation, without presuming a model for the Reynolds stresses (i. e. for both $k - \varepsilon$ and SMC) are written as:

$$\frac{dk}{dt} = -R_{11}S_{11} - \varepsilon \quad (4)$$

$$\frac{d\varepsilon}{dt} = (-c_{\varepsilon 1}R_{11}S_{11} - c_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k} \quad (5)$$

where R_{11} stands for the turbulence stress component $\overline{u_1 u_1}$. Standard values of the model coefficients are $c_{\varepsilon 1} = 1.44$ and $c_{\varepsilon 2} = 1.92$. The equation for the turbulent inverse time-scale $f = \varepsilon/k$ is readily obtained from the previous:

$$\frac{f'}{f} = -\frac{R_{11}}{k}S_{11}(c_{\varepsilon 1} - 1) - (c_{\varepsilon 2} - 1)f \quad (6)$$

Introducing the anisotropy of the normal stress, $a(t) = R_{11}(t)/k(t) - 2/3$, yields:

$$\frac{f'}{f} = -(2/3 + a)(c_{\varepsilon 1} - 1)S - (c_{\varepsilon 2} - 1)f \quad (7)$$

Explicit solutions for algebraic stress models are given by Gatski and Speziale 1993, but only for constant strains. Cambon and Scott 1999 produced an analytic solution to the present problem, using Rapid Distortion Theory and a clever rescaling of all variables including time, but RDT requires $f << S$, which is not what the present solution will show. Though RDT makes less modelling assumptions than SMC, it cannot be deduced from RDT that turbulence must decay on the long term, since RDT is limited to short times after the initial condition that turbulence is already "weak" compared to the strain. The objective here is to attempt to understand (limiting ourselves to simpler algebra than in Cambon and Scott 1999) why standard numerical simulations shown further lead to systematic increase of turbulence kinetic energy when using the EVM, and systematic decay with the second moment closure.

QUASI ISOTROPIC TURBULENCE

When anisotropy a is assumed neglegible before 2/3 in (7), the inverse turbulent time-scale is implicitly given by:

$$\frac{f(t)}{f_0} = \exp\left(-(c_{\varepsilon 2} - 1)\int_0^t f(\tau)\, d\tau\right)\left(1 + \frac{4}{5}\sin(\omega t)\right)^{-n} \quad (8)$$

Where $n = 2/3(c_{\varepsilon 1} - 1)$. Another form of solution suggested by a Reviewer, using the Floquet theory (Bender and Orszag 1978) and significant algebra is:

$$f(t) = \frac{f_0\left(1 + \frac{4}{5}\sin(\omega t)\right)^{-n}}{1 + f_0(c_{\varepsilon 2} - 1)\int_0^t\left(1 + \frac{4}{5}\sin(\omega\tau)\right)^{-n} d\tau} \quad (9)$$

Neither solution is explicit, but they take the form of a periodic function with a quasi exponential decay $D(t)$ of the amplitude:

$$f(t) = f_0 D(t)\left(1 + \frac{4}{5}\sin(\omega t)\right)^{-n} \quad (10)$$

The objective in this section is to explain the numerical simulation results presented further, hopefully without numerical integration. Replacing $f(t)$ in the integral, $\int_0^t f(\tau)\, d\tau$, by its initial value, f_0 leads to the approximation $D(t) \approx \exp(-(c_{\varepsilon 2} - 1)f_0 t)$. This is admittedly a rather crude simplification corresponding to $(c_{\varepsilon 2} - 1)f^2 \approx (c_{\varepsilon 2} - 1)f f_0$ in the initial ODE, thus overestimating the damping by the dissipation term, but nevertheless it allows to give a qualitative illustrations of the behavior of f, which will be confirmed by the classical numerical solutions shown further.

For an intermediate value of the strain parameter, $\eta_0 = S_0/f_0 = 1$ (initial turbulent time-scale=piston period) we obtain the result in figure

2 where the turbulence has virtually vanished in about 5 cycles.

For a weak strain, typically $\eta_0 = S_0/f_0 = 0.1$, the turbulence decay dominates, and turbulence vanishes before the cycle is complete. The above assumptions are most valid for strong strains, e.g. $\eta_0 = S_0/f_0 = 10$: the anisotropy has not enough time to follow the cyclic strain-rate, and $f(t) \simeq f_0$ is legitimate. The analytical solution is plotted on figure 3.

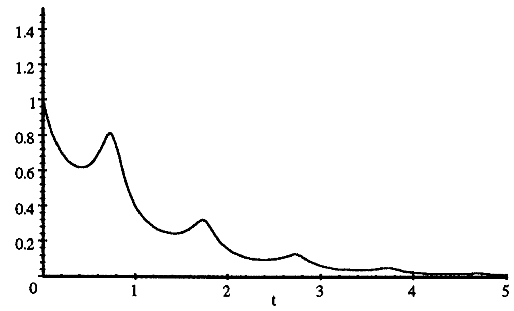

Figure 2. Inverse timescale in quasi isotropic turbulence for moderate strain rate parameter $\eta_0 = 1$.

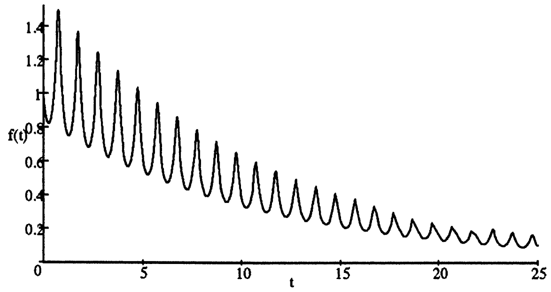

Fig. 3. Inverse timescale in quasi isotropic turbulence for strong strain rate parameter $\eta_0 = 10$.

From the above we conclude that when the Reynolds stress tensor is assumed isotropic, turbulence can only be sustained through a number of periods in case of strong strains, or rapid distortion, but will eventually decay.

We now move to anisotropic cases and show first that, when taking anisotropy into account, the use of the eddy viscosity (EVM) assumption is erroneous. In the second moment closure (SMC) framework, which has the advantage of using the exact production terms, the turbulent frequency can only decrease, while in the eddy viscosity (EVM) framework, this frequency increases and scales with S.

ENERGY BALANCE OVER A CYCLE

Let $S_{ij}(t)$ be any imposed homogeneous cyclic strain-rate, we then seek the variation of the turbulence kinetic energy over one cycle. We can write the energy rate equation in a general form for homogeneous turbulence:

$$\frac{dk}{dt} = -R_{ij}S_{ij} - \varepsilon \tag{11}$$

$$R_{ij} = (a_{ij} + \frac{2}{3}\delta_{ij})k \tag{12}$$

S_{ij} is periodic, and in the long term, so is a_{ij}. The rate of change of turbulent kinetic energy is obviously overestimated when we neglect dissipation, as seen from the following analysis. The energy rate equation can be written in terms of stress anisotropy and mean rate of strain:

$$\begin{aligned}
\frac{dk}{dt} &= -(a_{ij} + \frac{2}{3}\delta_{ij})kS_{ij} - \varepsilon \\
\frac{dk}{kdt} &= \frac{d\ln(k)}{dt} = -(a_{ij} + \frac{2}{3}\delta_{ij})S_{ij} - \frac{\varepsilon}{k} \\
\ln(\frac{k(T)}{k(0)}) &= -\int_0^T a_{ij}S_{ij}dt - \frac{2}{3}\left[I_{ij}\right]_0^T - \int_0^T \frac{\varepsilon}{k}dt
\end{aligned}$$

where

$$\frac{2}{3}\left[I_{ij}\right]_0^T = \int_0^T \frac{2}{3}\delta_{ij}S_{ij}dt = 0 \tag{13}$$

since the cumulated strain I_{ij} is periodic.

The isotropic part of the Reynolds stress tensor contributes to zero net production over one cycle. The growth over a period, whatever the turbulence model (no modelling assumption needed at this point), is:

$$\ln(\frac{k(T)}{k(0)}) = \int_0^T \left[(-a_{ij}S_{ij}) - \frac{\varepsilon}{k}\right] dt \tag{14}$$

EDDY VISCOSITY MODELS

Obviously, the EVM, whereby the stresses are always colinear with the strain-rate, introduces maximum production, i. e. the EVM assumption $a_{ij} = -2\nu_t S_{ij}/k$ leads to:

$$\ln(\frac{k(T)}{k(0)}) = \int_0^T (2\frac{\nu_t}{k}\left\|S\right\|^2 - \frac{\varepsilon}{k})dt \tag{15}$$

Introducing the $k - \varepsilon$ model for the eddy viscosity:

$$\ln(\frac{k(T)}{k(0)}) = \int_0^T (2C_\mu \frac{S^2}{f^2} - 1)f dt \tag{16}$$

Thus according to the $k - \varepsilon$ model (or any other EVM), for weak initial turbulence or strong strain, k increases, and there presumably exists a steady solution with an average over one cycle for:

$$<\eta>_T = <\frac{S}{f}>_T \approx \frac{1}{\sqrt{2C_\mu}} = 2.35 \tag{17}$$

APPROXIMATE ANALYTIC SOLUTIONS TO THE SMC

The present investigation was initiated by the full SMC numerical simulations of in-cylinder flow within a European Joule project (Hanjalić et al 1999), which invariably lead to decay of turbulence when a complete but valveless cylinder-piston assembly was considered. The analysis in the previous section seems to explain this fact. The standard second-moment closure, where the pressure-strain term is modelled by the Rotta's linear return to isotropy model for the slow part and the isotropization of production model (IP) for the rapid part, and in the case of homogeneous cyclic compression, reduces to:

$$\frac{dR_{11}}{dt} = -(1-C_2)2R_{11}S - aC_1\varepsilon - \frac{2}{3}\varepsilon \qquad (18)$$

with $C_1 = 1.8$ and $C_2 = 0.6$.

Using the relations: $a(t) = \frac{R_{11}(t)}{k(t)} - \frac{2}{3}$; $a' = \frac{R_{11}'}{k} - \frac{R_{11}}{k^2}k'$; $\frac{dk}{dt} = -R_{11}S - \varepsilon$, the equation for the anisotropy $a(t)$ is obtained:

$$a' = -p_a S - (C_1 - 1)af \qquad (19)$$

$$p_a = \left[\frac{4}{3}(\frac{2}{3} - C_2) + (\frac{2}{3} - 2C_2)a - a^2\right] \qquad (20)$$

Plotting the parabola p_a shows it can be linearized for $-0.1 < a < 0.1$. Next, reducing the Rotta constant to, $C_1 = 1.$, instead of 1.8, conveniently cancels the af term. Thus, retaining the first two terms, i.e:

$$a' = -\frac{4}{3}(\frac{2}{3} - C_2)S + (-\frac{2}{3} + 2C_2)Sa \qquad (21)$$

leads to the solution:

$$a = \frac{A(\sin(\omega t) + \frac{5}{4})^{(-\frac{2}{3}+2C_2)} + \frac{4}{3}(\frac{2}{3} - C_2)}{(-\frac{2}{3} + 2C_2)} \qquad (22)$$

which is shown on figure 4, where the anisotropy remains moderate, i.e. between -.0.06 and 0.1 , in agreement with the hypothesis.

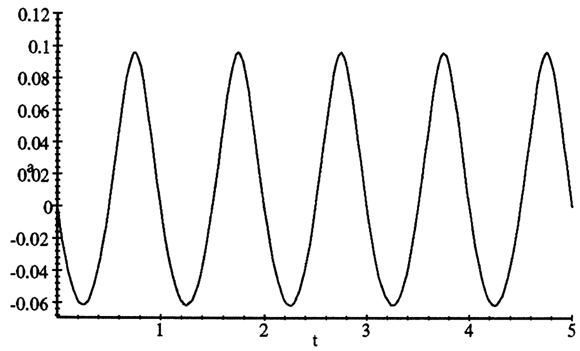

Figure 4. Stress anisotropy in response to the production by mean rate of strain (including rapid pressure-strain term), with $C_1 = 1.$ and neglect of Sa^2.

Furthermore note that this is actually an overestimation of the magnitude of a since the return to isotropy constant (Rotta) has been significantly reduced. This also justifies the assumption that the production term in the k equation can be written as $(a+2/3)S \approx 2/3S$, i.e. the "quasi isotropic turbulence" assumption leading to the variation of f given in figure 3 is confirmed.

CONJECTURE

From equation (14), only the anisotropic part may contribute to increasing or sustaining the turbulence energy over a cycle:

$$\ln(\frac{k(T)}{k(0)}) = \int_0^T [(-aS) - f]\, dt \qquad (23)$$

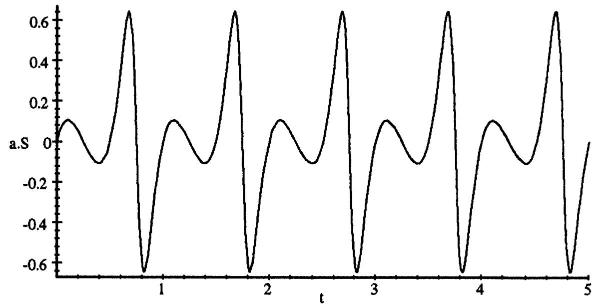

Figure 5. Production term $(-aS)$ with a from equation (22)).

Figure 5 shows the cyclic variation of the product $(-aS)$, using the solution for a, eq. (22), with obvious zero period integral. Indeed:

$$\int_0^T a\, S\, dt = \int_0^T (\sin(\omega t) + \frac{5}{4})^{-m}\omega\cos(\omega t)dt = 0 \qquad (24)$$

with $m = \frac{2}{3} + 2C_2 - 1$ so:

$$ln(f(T)/f(0)) = -\int_0^T f.dt < 0 \qquad (25)$$

Hence one can 'Conjecture', that in the SMC framework, when assuming a periodic solution for all variables, production over one cycle is zero. Thus over a long period of time the turbulence can only decay.

This conjecture is not formally proven here, but assuming periodic variations for the anisotropy, strain, and cumulated strain, a_{ij}, S_{ij}, I_{ij} leads to:

$$-\int_0^T a_{ij}S_{ij}dt = -\left[a_{ij}I_{ij}\right]_0^T + \int_0^T \frac{da_{ij}}{dt}I_{ij}dt$$

The first term $[a_{ij}I_{ij}]_0^T$ is zero, and a model can be inserted in place of $\frac{da_{ij}}{dt}$, but it remains non trivial to show that this second term amounts to zero, unless $\frac{da_{ij}}{dt}$ is directly proportional to the strain, with constant proportionality coefficients. A similar result was exhibited by Cambon et al., using rapid distortion theory assumptions, but the conjecture seems to be more general. The RDT hypotheses are not compatible with the fact that if a periodic solution is assumed, the turbulence frequency must have the same ordre of magnitude as the strain for production to balance dissipation in equation (23).

NUMERICAL SOLUTIONS WITH COMPLETE MODELS

Classical numerical computations are now presented, using both the SMC and EVM, with no other simplification than homogeneity. In fact these computations were performed first, then the previous analytical study was undertaken to explain why all SMC simulations led to decay of turbulence. The computation was performed for a realistic situation for a cylinder diameter $D = 100$ mm and stroke $L = 100$ mm, with rotation rate of 2000 rpm, using air as fluid ($\nu = 1.5 \times 10^{-5}$ m^2/s). Adopting the sinusoidal variation of piston velocity over the stroke L, $U_p = U_0\, sin(\omega t)$, the maximum piston velocity is $U_0 = 10.47$ m/s.

Figures (6) and (7) show in parallel the typical results obtained with the SMC and EVM for the same initial conditions. Presented are the evolution of the turbulence stress components $\overline{u_i u_j}$ over a series of cycles, and of the budget of k for a selected characteristic single cycle.

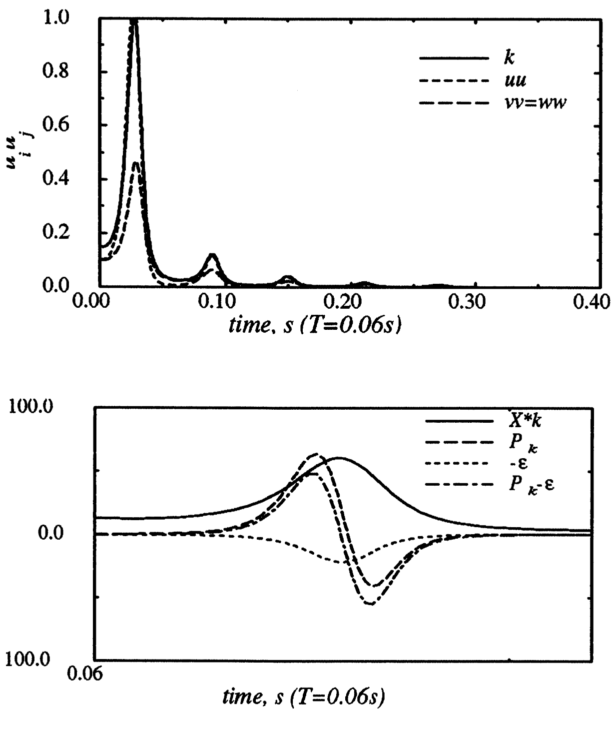

Fig. 6. The SMC computations: (top) Evolution of Reynolds stresses ; (bottom) Blow up over a single cycle of the Budget of kinetic energy (magnified by X factor to fit the figure frame).

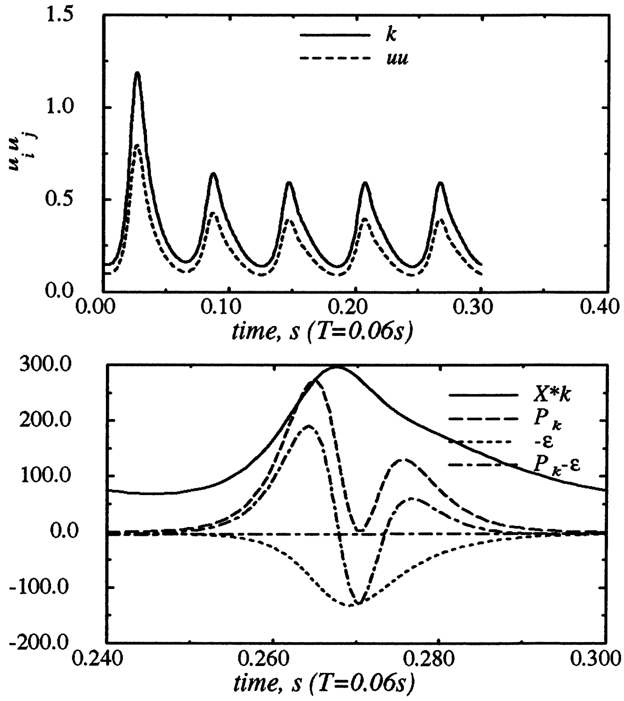

Fig. 7. Same as fig. 6, but for the EVM.

The results obtained with the SMC were obtained using Rotta model of the slow part of the Φ_{ij} term and IP model of the rapid term, but any other model would give similar behaviour and the same outcome over a single period. From the model computations we observe that the stresses increase during the compression stroke and reach the peak values at the top dead center with a change of sign of the kinetic energy production. In contrast, the integral of P_k^{EVM} over a period T is always positive, as follows from the expression (15). Together with ε it establishes a balance in time after some cycles as in (17), which is erroneous.

COMMENTS ON THE EFFECT OF THE WALL SHEAR

Although the above analysis is performed for homogeneous flows away from a solid wall, it reveals several important facts. The most important outcome is that a periodic compression/expansion can not alone generate the turbulence. Yet in a typical IC engine the flow is highly turbulent. Other major sources of the in-cylinder turbulence are: the shear in the fuel jets and air jets issuing from the valves, swirl, tumble and other secondary motions generated on purpose by adequate cylinder-piston geometry, and the wall shear layer.

The latter effect can be estimated from a qualitative analysis of oscillating boundary layers. It is known that such boundary layers will become turbulent (at least during a part of a cycle) only if the Reynolds number $Re_{\delta_S} = U_0 \delta_S / \nu$, based on the Stokes thickness $\delta_S = \sqrt{2\nu/\omega}$, exceeds the critical value of about 600. Here U_0 is a reference free stream velocity, and the Stokes thickness is a measure of the "penetration depth" in an oscillating flow. For a typical situation in an IC engine, assuming that U_0 can be identified with the maximum piston velocity and evaluating ω for a typical rotation speed, Re_{δ_S} (here, for the case considered, ≈ 264) is much smaller than the critical value so that the wall shear layer has no significant effect on the turbulence generation. Hence, the only remaining sources of turbulence are the fuel and air jets and the large-scale vortical motions in the cylinder.

CONCLUSIONS

Both analytical and numerical integrations of the second moment closures (SMC) for the Reynolds stresses, tend to show that cyclic straining as in IC engines, approximated by homogeneous strains, yields zero net production of energy over one cycle. In fact, due to a steady decay in turbulent stress because of viscous dissipation, the production diminishes from cycle to cycle, and turbulence is never sustained. This conjecture supported by many simulations, and proven elsewhere using rapid distortion hypotheses, was however not formally proven herein.

In fact, if one assumes production-dissipation equilibrium, then the turbulence frequency needs to be similar to the strain rate, in which case one is remote from the rapid distortion theory. On the other hand it seems reasonable that non linear terms providing lower anysotropy and introducing dissipation, should lead to lower overall production than the RDT, thus supporting the conjecture.

The eddy viscosity models (EVM), because of the stress-strain systematic alignement they assume, obviously lead to an artificial generation of energy through the compression-expansion cycles, and should not be used in such flows. "Dissipation modifications" to the EVM model of the RNG type or other, although reducing the energy, do not seem to correctly address the problem which clearly come from the production in cyclic strains.

An accurate computation of flow and turbulence in an IC engine within the framework of single-point statistical modelling requires the second-moment (Reynolds-stress) closure level, since any eddy-viscosity approach leads to a drastic overestimation of turbulence production.

The complete inhomogeneous SMC simulation of the the flow in a valveless cylinder-piston assembly ([?]oulrp99) shows a very low level of small scale turbulence, confined to the near-wall region, hardly diffusing toward the center of the chamber. Moreover this flow region remains turbulent for only very high Reynolds numbers (large cylinders and rotation speeds), which may exist only in large real life IC engines, while in standard size car-engine the turbulence produced by wall boundary layer is insignificant. A qualitative analysis shows that the typical Reynolds number based on Stokes thickness is far below the critical value, which would ensure a sustenance of wall generated turbulence in an oscillating wall boundary layer.

While in any valves piston-cylinder assembly a weak secondary motion will usually appear as a consequence of wall friction and piston movement, this motion is too weak to contribute significantly to turbulence production. Hence, the only source of turbulence are the fuel and air jets and cavities in piston or cylinder head.

The latter two conclusion indicates that an idealized valveless piston-cylinder assembly, as often used in the experimental research of engine flow, is of little relevance to the study of turbulence in IC engines.

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