Numerical simulation of forced two–dimensional turbulence using wavelets

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Abstract

This paper presents a numerical simulation of statistically stationary two-dimensional turbulence. Navier–Stokes equations are integrated in an adaptive wavelet basis where only the evolution of significant coefficients is computed. The forcing of the flow is also done in wavelet space with enstrophy being injected in both space and scale. The results show that the flow has reached a statistically stationary state, which is proved by the fact that energy and enstrophy remain constant during the flow evolution while the energy spectrum and the PDF of vorticity are also maintained. This new forcing, defined in wavelet space, allows to model the local production of vortices by instabilities as generically observed in turbulent shear flows.

1 Introduction

The formation of coherent vortices characterizes the nonlinear dynamics of turbulent flows, in particular shear flows. This leads to a sparse representation in wavelet bases. We have shown [4] [5] that in two–dimensional turbulent flows the coherent vortices correspond to very few wavelet coefficients: the strongest ones contain most of the enstrophy, while the weaker coefficients represent the unorganized background flow . Therefore the wavelet representation is an efficient basis to study two-dimensional turbulent flows, since the dynamics of such flows is largely controlled by their coherent vortices. Wavelets have first been used for analysis or compression of turbulent flows [4] and, more recently, we have developed a numerical scheme to compute two-dimensional turbulent flows directly in an orthogonal wavelet basis using a Petrov–Galerkin scheme [7], [8], [12]. This scheme dynamically adapts the basis to follow the flow evolution in both space and scale, with the nonlinear term being computed on a locally refined grid [13]. A different approach has been developed in [2], which uses wavelets to compress the matrix–vector operations resulting from a finite difference discretization of the Navier–Stokes equations. Both schemes have been validated by calculating the nonlinear interaction of three vortices and comparing the results with those of a classical pseudo–spectral code.

We have then applied our scheme to compute a freely decaying turbulent flow [7] and a temporally developing mixing layer [15]. For both cases the wavelet representation allows to reduce the number of degrees of freedom to be computed, although these fully developed turbulent flows exhibit a large number of spatial scales.

To obtain statistically stationary turbulent regimes, classically two different forcing schemes are used [1], which both operate in Fourier space:

(i) a negative dissipation within a given wavenumber band, with an amplification coefficient which depends on the wavenumber,

(ii) a white or coloured noise in time, with a prescribed isotropic spectral distribution, strongly peaked in the vicinity of a given wavenumber, with random phases.

For both schemes the choice of the wavenumber band represents that part of the energy spectrum where instabilities have a significant growth rate. But none of these Fourier forcings are satisfactory to model instabilities (e.g. the Kelvin–Helmholtz instability encountered in shear flows), because they inject energy and enstrophy locally in Fourier space and therefore non-locally in physical space. The forcing affects all spatial locations in a homogeneous fashion, while on the contrary instabilities excite vortices locally in space. Another drawback of Fourier forcing is that the scale of the coherent vortices is imposed by the scale of the forcing.

In [11] we have developed a new method to force turbulent flows by local vortex excitation, where the forcing term depends nonlinearly on the wavelet coefficients of the vorticity field. It injects energy and enstrophy as locally as possible, in both physical and spectral spaces, while controlling the smoothness of the vortices thus excited to avoid creating any unphysical discontinuities in the vorticity field.

The purpose of the present paper is the simulation of wavelet forced two-dimensional turbulent flows using an adaptive wavelet basis. Hence we couple the wavelet based forcing method [11] with the fully adaptive wavelet code [13] [15] and report on numerical results obtained.

2 Governing equations and wavelet forcing

To numerically simulate forced two-dimensional turbulence we consider the Navier–Stokes equations written in velocity–vorticity form with a forcing term F. Furthermore we include an artificial dissipative term $\lambda\Psi$, a so-called Rayleigh friction [1], which provides an energy sink at large scales. This is necessary because the energy injected by external forcing tends to accumulate at large scales (due to the inverse energy cascade characteristic of two-dimensional turbulent flows) and should therefore be dissipated there to maintain a statistically stationary regime. The governing equations are :

$$\partial_t\omega + \mathbf{v}\cdot\nabla\omega = \nu\nabla^2\omega + \lambda\Psi + F\ ,\ \nabla\cdot\mathbf{v} = 0 \quad (1)$$

with the velocity field $\mathbf{v} = (u,v)$, the vorticity $\omega = \nabla\times\mathbf{v}$, the stream function $\Psi = \nabla^{-2}\omega$ and the kinematic viscosity ν. In order to simulate turbulent flows far from the wall regions and to avoid the treatment of boundary layers, we assume periodic boundary conditions in both directions, i.e. our domain is the two-dimensional flat torus $\mathbb{T}^2$ with $\mathbb{T} = 2\pi\mathbb{R}/\mathbb{Z}$.

Here we apply a nonlinear wavelet based forcing approach, which is triggered directly by the nonlinear dynamics of the flow. We define the forcing term F as a function of ω, reconstructed from a subset of its wavelet coefficients $\tilde{\omega}$ using a two-dimensional multiresolution analysis (MRA) [3] [4]:

$$F(x,y) = C \sum_{J_0<j<J_1} \sum_{k_x=0}^{2^j-1} \sum_{k_y=0}^{2^j-1} \sum_{\mu=1,2,3} \tilde{\omega}^{\mu}_{j,k_x,k_y}\, \psi^{\mu}_{j,k_x,k_y}(x,y) \quad (2)$$

with $0 \le J_0 \le J_1 \le J$, where J denotes the finest level in the simulation, $C > 0$ and $|\tilde{\omega}^{\mu}_{j,i_x,i_y}| > \epsilon$. The scale parameters J_0 and J_1 define the scale range of the forcing, from the largest scale 2^{-J_0} to the smallest scale 2^{-J_1}. The restriction to the wavelet coefficients above a given threshold ϵ implies that only the dynamically active part of the flow, i.e. the coherent vortices, are forced [11]. Due to orthogonality, the coefficients in eq. (2) are given by $\tilde{\omega}^{\mu}_{j,i_x,i_y} = \langle\omega\ ,\ \psi^{\mu}_{j,i_x,i_y}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. The two-dimensional wavelets ψ^{μ}_{j,i_x,i_y}, constituting a MRA, are defined as:

$$\psi^{\mu}_{j,i_x,i_y}(x,y) = \begin{cases} \psi_{j,i_x}(x)\ \phi_{j,i_y}(y) & ,\ \mu = 1 \\ \phi_{j,i_x}(x)\ \psi_{j,i_y}(y) & ,\ \mu = 2 \\ \psi_{j,i_x}(x)\ \psi_{j,i_y}(y) & ,\ \mu = 3 \end{cases} \quad (3)$$

where $\phi_{j,i}$ and $\psi_{j,i}$ are the 2π–periodic one-dimensional scaling function and the corresponding wavelet, respectively.

The strength of the forcing C and the Rayleigh friction λ are adjusted in such a way that we obtain a statistically stationary state.

3 Adaptive wavelet scheme

To solve numerically (1) we discretize it, first in time and then in space (contrarily to classical schemes, e.g. line methods), to be able to remap the basis, used for the spatial discretization, at each time step. For the time discretization we employ classical semi–implicit finite differences of second order, with an Euler backwards step for the viscous term, and an Adams–Bashforth extrapolation for the convection, friction and forcing terms:

$$\begin{aligned}(\sigma - \nu\nabla^2)\,\omega^{n+1} &= \frac{4}{3}\sigma\omega^n - \frac{1}{3}\sigma\omega^{n-1} \\ &- (\mathbf{v}^\star\cdot\nabla\omega^\star - \lambda\Psi^\star - F^\star) \quad (4)\end{aligned}$$

with time step Δt, $\sigma = 3/(2\Delta t)$ and where $f^\star = 2f^n - f^{n-1}$.

The space discretization is based on a method of weighted residuals. Therefore we develop ω^n into an orthonormal wavelet series

$$\omega^n(x,y) = \sum_{j=0}^{J-1} \sum_{i_x=0}^{2^j-1} \sum_{i_y=0}^{2^j-1} \sum_{\mu=1}^{3} \tilde{\omega}^{\mu,n}_{j,i_x,i_y}\ \psi^{\mu}_{j,i_x,i_y}(x,y) \quad (5)$$

and we apply a Petrov-Galerkin scheme to (4) with test-functions θ, being solutions of its elliptic linear part:

$$(\sigma - \nu\nabla^2)\,\theta^{\mu}_{j,i_x,i_y}(x,y) = \psi^{\mu}_{j,i_x,i_y}(x,y) \quad (6)$$

These functions are called vaguelettes [1] and have similar localization properties in scale and space as wavelets have [10]. They are translation invariant but, due to the inhomogeneity of the operator, the scale invariance is only recovered asymptotically, i.e. for $j \to \infty$. Furthermore the vaguelettes in (6) can be calculated analytically in Fourier space. Their values are calculated and stored in a preprocessing step of the algorithm.

By construction of the test functions the resulting stiffness matrix is diagonalized and therefore we avoid assembling and solving a linear system at each time step [8]. Hence the solution of (4) reduces to a change of basis

$$\begin{aligned} \tilde{\omega}^{\mu,n+1}_{j,i_x,i_y} &= \langle \frac{4}{3}\sigma\omega^n - \frac{1}{3}\sigma\omega^{n-1} \\ &- (\mathbf{v}^\star \cdot \nabla\omega^\star - \lambda\Psi^\star - F^\star) \,,\, \theta^\mu_{j,i_x,i_y}\rangle \end{aligned} \tag{7}$$

and the active wavelet coefficients of the solution $\tilde{\omega}^{n+1}$ at time step t^n are computed by means of a fast vaguelette decomposition (c.f. [8]). A reduction of the number of degrees of freedom is obtained by retaining only those coefficients with $|\tilde{\omega}^{\mu,n+1}_{j,i_x,i_y}| > \epsilon$, the same threshold as the one employed to define the forcing F (2). The index set of active coefficients in the next time step is then determined from the previous step by compression of $\tilde{\omega}$ with the required tolerance ϵ. In order to extrapolate the flow evolution in space and scale for the next time step we also add to this index set the indices of the active coefficients' neighbours. For more details we refer to [8].

The nonlinear term $\mathbf{v}^\star \cdot \nabla\omega^\star$ is computed by an adaptive collocation in physical space [13]. This method (also called pseudo–wavelet scheme) employs fast transforms between sparse coefficient sets and locally refined grids [8]. Furthermore a Poisson equation $\nabla^2 \Psi^\star = \omega^\star$ is solved using a Petrov–Galerkin scheme, as for (4). In this case the test functions θ are solutions of the Poisson equation $\nabla^2 \theta^\mu_{j,i_x,i_y}(x,y) = \psi^\mu_{j,i_x,i_y}(x,y)$. The wavelet coefficients of the stream function $\Psi^\star$ are then calculated using the fast vaguelette decomposition previously mentioned. Applying an inverse adaptive wavelet transform, the stream function is reconstructed on a locally refined grid. Subsequently, the velocity $\mathbf{v}^\star = (-\partial_y\Psi^\star, \partial_x\Psi^\star)$ and $\nabla\omega^\star$ are calculated using finite differences of 4th order on an adaptive grid. Then the scalar product $\mathbf{v}^\star \cdot \nabla\omega^\star$ can be calculated at the grid points. Finally, the right hand side of (7) is summed up on the adaptive grid in physical space and then the wavelet coefficients of the vorticity ω^{n+1} are calculated using the adaptive vaguelette decomposition.

[1] The term vaguelette is derived from the french term 'ondelette', which means wavelet.

4 Results

We present a numerical simulation of a wavelet forced two–dimensional turbulent flow computed in an adaptive wavelet basis. The finest scale accessible in this simulation is 2^{-8}, equivalent to the resolution of a non–adaptive scheme with 256^2 degrees of freedom. The computation is initialized using a vorticity field obtained from a decaying simulation just after the formation of coherent vortices. The statistically steady regime has been reached with the following parameters: the forcing scale range lies in between $J_0 = 3$ and $J_1 = 7$, viscosity $\nu = 9.9 \cdot 10^{-4}\,\mathrm{m^2s^{-1}}$, strength of the forcing $C = 2.5 \cdot 10^{-1}\,\mathrm{s^{-1}}$, threshold $\epsilon = 5 \cdot 10^{-5}\,\mathrm{s^{-1}}$ and Rayleigh friction coefficient $\lambda = 1\,\mathrm{m^{-2}s^{-1}}$. For the wavelet decomposition we use spline wavelets of order 4.

In Fig. 1 we observe that energy E and enstrophy Z are maintained without any oscillation, contrarily to Fourier forcing where E and Z oscillate around their mean values. Figure 2 displays the energy and enstrophy spectra at $t = 0\,\mathrm{s}$ and $20\,\mathrm{s}$. They maintain their shape with k^{-5} and k^{-3} power–law behaviours, respectively, during the whole computation. The slopes are therefore steeper than Kraichnan's prediction [9] as also observed with Fourier forcing [1].

The statistical stationarity of the flow is also reflected in the fact that the vorticity PDF does not change in time (cf. Fig. 3). This PDF exhibits a non–Gaussian behaviour, as observed in both numerical and laboratory experiments [17]. The heavy tails correspond to the coherent vortices which are responsible for the flow intermittency [16].

In Fig. 4 we plot the vorticity field at $t = 0\,\mathrm{s}$ and $20\,\mathrm{s}$. We observe that the strongest vortices are reinforced during the flow evolution. In particular strained vortices are rolling up, as enstrophy is locally injected into them by the wavelet forcing, in a way very similar to the rolling up of vorticity sheets by Kelvin–Helmholtz instability. We also check that the same–sign vortex merging mechanism, characteristic in two–dimensional turbulent flows, is not inhibited by the wavelet forcing. We find that vorticity at $t = 20\,\mathrm{s}$ exhibits less filaments in the background, due to the fact that at each time step the weak wavelet coefficients are discarded.

In Fig. 5 we display wavelet coefficients of vorticity. We use Mallat's representation [3] which shows the same field at different scales and for different directions (horizontal corresponding to $\mu = 1$

in eq. (3), vertical to $\mu = 2$ and diagonal to $\mu = 3$), with the wavelet coefficients $\tilde{\omega}^{\mu}_{j,i_x,i_y}$ plotted using a logarithmic scale. They are placed at $x = 2^j(1-\delta_{\mu,1})+i_x$, $y = 2^j(1-\delta_{\mu,2})+i_y$, δ being the Kronecker tensor, with the origin in the upper left corner and the y-coordinate oriented downwards. The largest scales correspond to the smallest square (top left on Fig. 5) while the smallest scales correspond to the largest squares (bottom left for the horizontal direction, top right for the vertical direction and bottom right for the diagonal direction). The scale repartition of the wavelet coefficients of vorticity confirm the strong intermittency, we have already noticed from the vorticity PDF. Actually, the representation of vorticity in wavelet space is a well suited diagnostics to characterize intermittency, because the increasing sparsity of the wavelet coefficients while scale decreases gives a quantitative measure of the flow intermittency [16].

In Fig. 6 we show the time evolution of the number of degrees of freedom used in the adaptive computation. It remains quasi-constant in time and represents 30% of the total number of coefficients necessary for a non–adaptive computation (i.e. 256^2 here). This compression is weak because the threshold we have chosen is very small ($\epsilon = 5 \cdot 10^{-5}\,\mathrm{s}^{-1}$), which implies that we have actually performed a DNS–like simulation. If we want to significantly increase the compression, we would then need a turbulence model in order to describe the effect of the discarded wavelet coefficients onto the retained ones. We have proposed such a model, called coherent vortex simulation (CVS) [6] and we plan to apply it to this wavelet forced problem in future work.

5 Conclusion and Perspectives

We have presented the time evolution of a wavelet forced two–dimensional turbulent flow computed in an adaptive wavelet basis. The results show that we have reached a statistically steady state, as the characteristic statistical diagnostics (i.e. total energy and enstrophy, the corresponding spectra and PDFs) do not change in time. The analysis of both vorticity PDF and the wavelet coefficients of vorticity reveals the flow intermittency, which is characteristic of the fully–developed turbulent regime. The evolution of vorticity shows that the wavelet forcing adequately simulates the Kelvin–Helmholtz instability which is generic to turbulent shear flows.

This wavelet forcing can also be applied to simulate inhomogeneous turbulent flows such as the two–dimensional temporally developing mixing layer [15], we have already computed in wavelets, but without forcing. Future work will be the wavelet computation of a forced mixing layer with the same forcing as presented here. Wavelet forcing is not limited to statistically homogeneous turbulence as it is the case for Fourier forcing, therefore it seems more physically sound.

Although the adaptive wavelet scheme and the wavelet forcing presented here have been designed for incompressible two–dimensional turbulent flows, they could be extended to three–dimensional turbulent flows, using the three dimensional vorticity-stream function formulation of the Navier–Stokes equation with vector valued wavelet and vaguelette transforms which is subject of current work.

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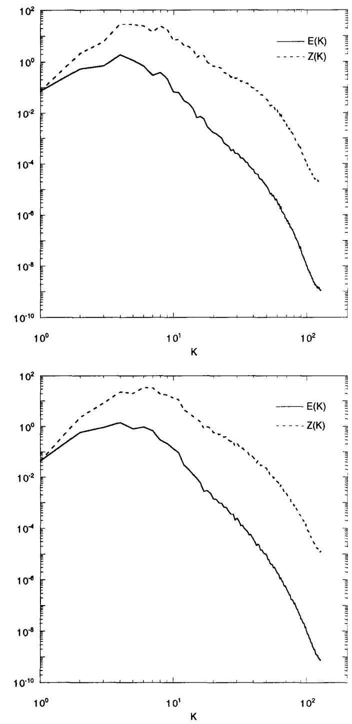

Figure 2: Energy/enstrophy spectra $t = 0$ and 20.

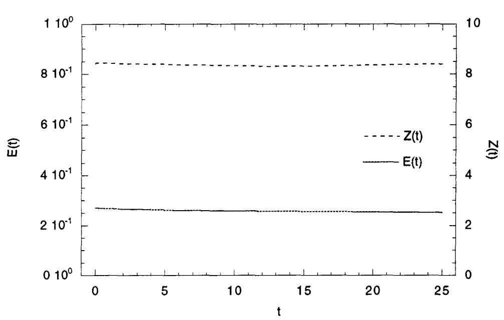

Figure 1: Evolution of energy $E(t)$ and enstrophy $Z(t)$.

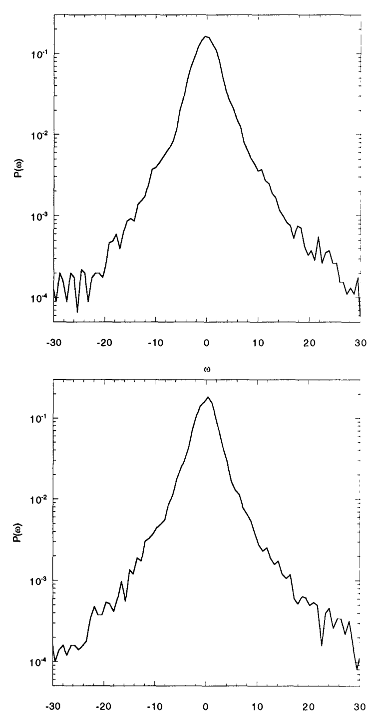

Figure 3: PDFs of vorticity at $t = 0$ and 20.

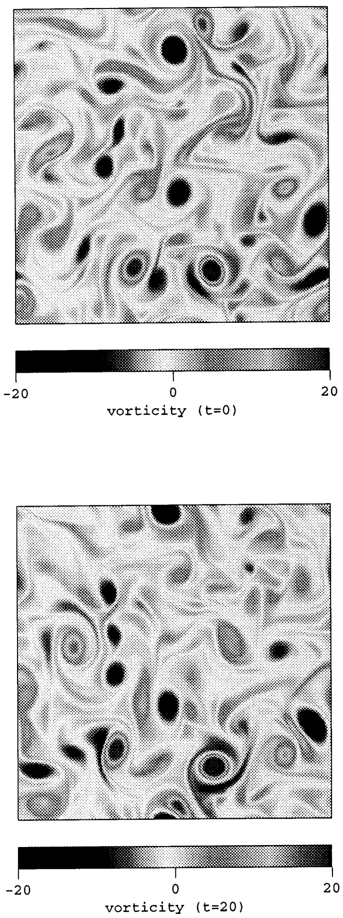

Figure 4: Wavelet forced 2D turbulent flow: vorticity field at $t = 0$ and 20.

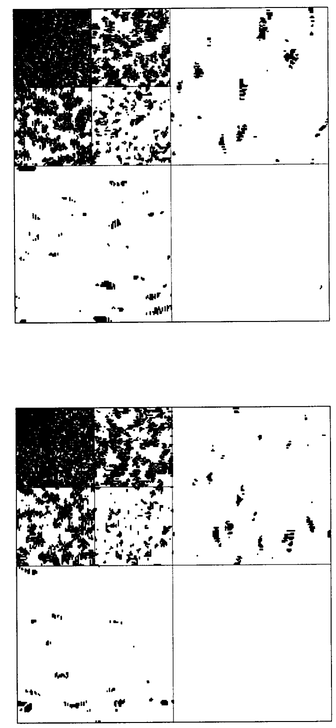

Figure 5: Corresponding active wavelet coeffcients (dark markers) at $t = 0$ and 20.

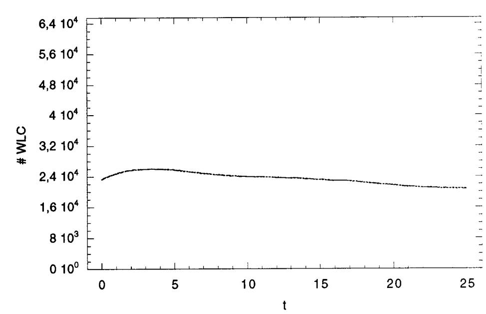

Figure 6: Evolution of number of active wavelet coefficients $\#WLC$.