TOWARDS A CALIBRATION OF THE LENGTH-SCALE EQUATION

Stéphane Catris **Bertrand Aupoix**
ONERA
Department for Models in Aerodynamics and Energetics
2 Avenue Edouard Belin
B.P. 4025
31055 TOULOUSE Cedex 4 FRANCE

ABSTRACT

A complete set of constraints is proposed to force a high Reynolds number turbulence model to correctly predict the boundary layer behaviour whatever the pressure gradient and the Reynolds number. The constraints are general and are presently applied to a two-equation model, using the Boussinesq assumption together with new forms of the inhomogeneous terms.

INTRODUCTION

For aeronautical applications, turbulence models are first asked to correctly predict wall values i.e. skin friction and wall heat flux. For high lift configurations, the response of the boundary layer to a positive pressure gradient and separation are key challenges.

Attention will be restricted here to incompressible flows. The velocity profile for a two-dimensional boundary layer, is plotted in wall variables in figure 1 where u_τ is the friction velocity $u_\tau = \sqrt{\tau_w/\rho}$ and ν the viscosity. The skin friction coefficient is directly related to the maximum value since $\frac{U_{ext}}{u_\tau} = \sqrt{\frac{2}{C_f}}$. For a zero pressure gradient boundary layer (ZPG), the wake is quite small and the logarithmic region extends with the Reynolds number. A good prediction of the skin friction coefficient whatever the Reynolds number requires that

- the near wall model provides the correct intercept for the logarithmic region,
- the slope of the logarithmic region is correct,
- the wake is well reproduced.

For accelerated flows (FPG), the wake slightly decreases so that the above constraints nearly guarantee the correct prediction of the skin friction coefficient. For strongly accelerated flows, the prediction of the relaminarization has to be provided by the near-wall model.

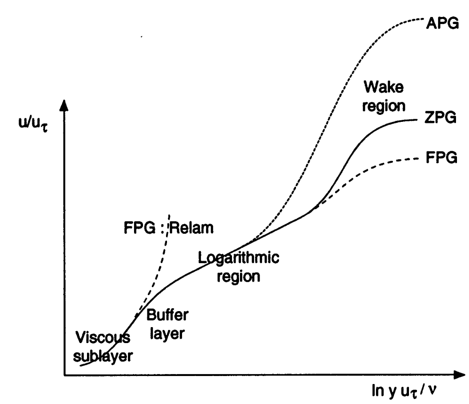

Figure 1. Boundary layer profile in wall variables for various pressure gradients

For decelerated flows (APG), the logarithmic region remains but decreases as the wake extends. The model has to predict the correct slope of the logarithmic region, again in order to provide good predictions whatever the Reynolds number, and to reproduce the large wake region.

Moreover, a model which is able to reproduce the wake of zero-pressure gradient and adverse pressure gradient flows is expected to correctly predict airfoil wakes as well as mixing layers which closely resemble these two flows.

Therefore, the strategy is to represent the evolution of the boundary layer through simple constraints and to determine the model constants from these constraints.

PROPOSED MODEL FORM

The paper will be restricted to two-equation models, using the eddy-viscosity assumption or Boussinesq relation. Moreover, only the high Reynolds number form of the model will be discussed, the near wall modelling will not be addressed.

With the Boussinesq assumption, the rôle of turbulence models is to determine the eddy viscosity, i.e. mainly to evaluate a turbulence velocity scale and a turbulence length scale. The turbulent kinetic energy transport equation, derived from the Navier–Stokes equation, requires little modelling and can provide the velocity scale. The length scale equation is more puzzling and is usually blamed for the model drawbacks.

For homogeneous flows, the turbulent kinetic energy transport equation requires no modelling

$$\frac{Dk}{Dt} = P_k - \varepsilon \tag{1}$$

P_k stands for the turbulent kinetic energy production rate and ε for the dissipation. The standard form of the dissipation transport equation

$$\frac{D\varepsilon}{Dt} = (C_{\varepsilon_1} P_k - C_{\varepsilon_2}\varepsilon)\,\frac{\varepsilon}{k} \tag{2}$$

yields predictions in good agreement with Roger's et al. (1986) DNS. Moreover, the transport equation for any length-scale determining quantity $\phi = k^m \varepsilon^n$ can be deduced from the above equations and reads

$$\frac{D\phi}{Dt} = \left(C_{\phi_1} P_k - C_{\phi_2}\varepsilon\right)\frac{\phi}{k}$$

The problem is thus to model the extra terms due to inhomogeneity. The exact form of the turbulent kinetic energy transport equation shows that the extra term is a divergence. Therefore, following Yoshizawa (1985), the turbulent kinetic energy transport equation is modelled as

$$\frac{Dk}{Dt} = P_k - \varepsilon + \operatorname{div}\left[\frac{\nu_t}{\sigma_{kk}}\,\underline{\operatorname{grad}}\,k + \frac{\nu_t}{\sigma_{k\varepsilon}}\,\frac{k}{\varepsilon}\,\underline{\operatorname{grad}}\,\varepsilon\right] \tag{3}$$

The length scale equation has been written a priori for $\phi = k^m \varepsilon^n$. In a way similar to the k equation, the diffusion flux involves two terms. There is no proof from the exact equation that inhomogeneous terms are of divergence form. Moreover, it is well known that when a length scale equation is expressed in terms of another length scale determining variable, e.g. when the ε-equation is written in terms of ω, dot products of gradients appear. Therefore, the following form is proposed for the length scale equation

$$\frac{D\phi}{Dt} = \left(C_{\phi_1} P_k - C_{\phi_2}\varepsilon\right)\frac{\phi}{k} + \operatorname{div}\left[\frac{\nu_t}{\sigma_{k\varepsilon}}\,\frac{\varepsilon}{k}\,\underline{\operatorname{grad}}\,k + \frac{\nu_t}{\sigma_{\varepsilon\varepsilon}}\,\underline{\operatorname{grad}}\,\varepsilon\right] \tag{4}$$
$$+\alpha\frac{\nu_t}{\phi}\,\underline{\operatorname{grad}}\,\phi\cdot\underline{\operatorname{grad}}\,\phi + \beta\frac{\nu_t}{k}\,\underline{\operatorname{grad}}\,k\cdot\underline{\operatorname{grad}}\,\phi + \gamma\frac{\nu_t\phi}{k^2}\,\underline{\operatorname{grad}}\,k\cdot\underline{\operatorname{grad}}\,k$$

It can be easily checked that, starting from (3, 4), the transport equation for another length scale determining variable $\psi = k^a \varepsilon^b (b \neq 0)$ can be deduced. This new equation has the same form as (4) and the coefficients $(\sigma, C, \alpha, \beta, \gamma)$ of the ψ-equation can be directly related to the ones in the k and ϕ equations. It can be mentioned that σ_{kk} changes with the length scale determining variable.

As the set of equations (3, 4) can be used for any length-scale determining variable, ε will be chosen from now to express the various constraints the model has to satisfy. The final choice of the length scale determining variable will be dictated by numerical stability arguments. On the one hand, dot products of gradients may lead to numerical stiffness and a change of variable allows to get rid of some of them. On the other hand, only the high Reynolds number form is addressed here; near-wall treatment will favour length scale determining variables which have a fair behaviour in the wall region.

At last, the model has to be completed with the expression for the eddy viscosity ν_t

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{5}$$

CONSTRAINTS

Strategy

As mentioned above, the strategy to determine coefficients is to look for all the constraints the model has to satisfy. As these constraints are simplified representations of the physics, their pertinence has to be checked. Therefore, the constraints have been derived for the general model (3, 4, 5) which involves all classical turbulence models such as $k-\varepsilon$, $k-\omega$, $k-\varphi$ or $k-L$ models. Thus, it has been possible to check the pertinence of these constraints by comparing the behaviour of the models and the expected behaviour according to the constraints. The constraints for the general model are quite complex and can be found in Catris and Aupoix

(1999). Only simplified expression for the $k-\varepsilon$ model will be given here for the sake of clarity.

Isotropic decay

The simplest constraint is the classical decay of isotropic turbulence which yields the bounds (Aupoix, 1987) $1.7 \leq C_{\varepsilon_2} \leq 2$.

Logarithmic region for a zero pressure gradient boundary layer

For a zero pressure gradient boundary layer, the total (laminar + turbulent) shear stress is constant in the near-wall region. In the logarithmic region, the laminar shear stress is negligible. Although its validity in the logarithmic region is questionable, Bradshaw's assumption is classically used to link the turbulent kinetic energy to the turbulent shear as $-<u'v'>=u_\tau^2=2a_1k$. Using wall scaling, i.e. making quantities dimensionless with u_τ and ν, it yields $k^+=\frac{1}{2a_1}$. For standard models in which $\sigma_{k\varepsilon}\to\infty$, neglecting advection yields an equilibrium between production and dissipation in the k-equation (3) which gives $\varepsilon^+=\frac{1}{\kappa y}$ where κ is the slope of the logarithmic region. This equilibrium still holds if $\sigma_{k\varepsilon}$ is finite. Substituting into the dissipation equation leads to

$$\left[\frac{(C_{\varepsilon_2}-C_{\varepsilon_1})\,2a_1}{\kappa^2}-\alpha\right]\sigma_{\varepsilon\varepsilon}=1 \qquad 2a_1=\sqrt{C_\mu} \tag{6}$$

Logarithmic region for a boundary layer with moderate pressure gradient

Experiments show that the slope of the logarithmic region is unchanged in presence of moderate pressure gradients. From the momentum equation, the dimensionless shear stress now reads

$$-<u'v'>^+=1+p^+y^+ \qquad p^+=\frac{\nu}{\rho u_\tau^3}\frac{dp}{dx} \tag{7}$$

Following Huang and Bradshaw (1995), all quantities are expanded in terms p^+ as

$$k^+=k_0^++p^+k_1^+\,;\,\varepsilon^+=\varepsilon_0^++p^+\varepsilon_1^+\,;\,\kappa^+=\kappa_0^++p^+\kappa_1^+y^+$$

The analysis only holds for moderate pressure gradients i.e. when $|p^+y^+|\ll 1$. Because of the form of the transport equations (3, 4), the analysis is somewhat more tedious than in Huang and Bradshaw. The zero pressure gradient boundary layer case investigated above is retrieved as zeroth order solution.

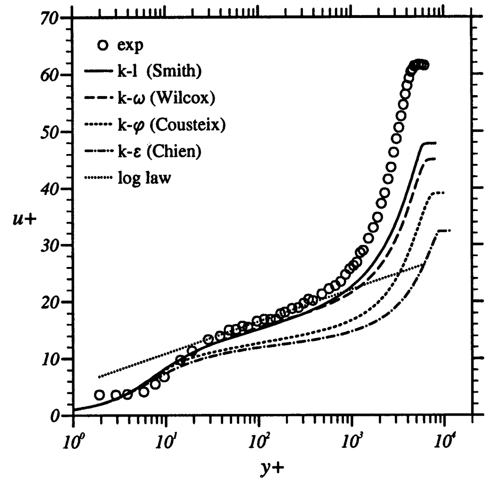

Figure 2. Prediction of the logarithmic region by various turbulence models for the positive pressure gradient experiment of Skåre and Krogstad

The first order solutions have the following form

$$\begin{aligned}
k_1^+ &= Ay^+ \quad ; \quad \varepsilon_1^+=\frac{1}{\kappa}\left(4a_1A-1-\frac{\kappa_1}{\kappa}\right)\\
A &= 2\frac{\dfrac{\kappa_0\,(\kappa_0+\kappa_1)}{2a_1\sigma_{k\varepsilon}}-1}{\kappa_0^2\left(\dfrac{1}{\sigma_{k\varepsilon}}+\dfrac{1}{\sigma_{kk}}\right)-4a_1}\\
\kappa_1 &\sim \left([2C_{\varepsilon_2}-C_{\varepsilon_1}+(\alpha+\beta)\,\sigma_{kk}]\,\alpha\sigma_{\varepsilon\varepsilon}^2\right.\\
&\quad +[(2\alpha+\beta)\,\sigma_{kk}+C_{\varepsilon_2}]\,\sigma_{\varepsilon\varepsilon}+\sigma_{kk})\,\sigma_{k\varepsilon}\\
&\quad +[(2\alpha+\beta)\,C_{\varepsilon_1}-(3\alpha+\beta)\,C_{\varepsilon_2}]\,\sigma_{kk}\sigma_{\varepsilon\varepsilon}^2\\
&\quad +(C_{\varepsilon_1}-2C_{\varepsilon_2})\,\sigma_{kk}\sigma_{\varepsilon\varepsilon}
\end{aligned} \tag{8}$$

where κ_1 should be null.

Figure 2 shows an example of the constraint validation. Predictions of Smith (1995) $k-L$ model, Wilcox (1988) $k-\omega$ model, Cousteix et al. (1997) $k-\varphi$ model and Chien's (1982) $k-\varepsilon$ model are compared for the equilibrium boundary layer case investigated by Skåre and Krogstad (1994). The $k-\omega$ and $k-L$ models which yield small values of κ_1 fairly reproduce the logarithmic region while the $k-\varepsilon$ or $k-\varphi$ models, which yield large values fail.

Square root region

For boundary layers submitted to strong positive pressure gradients, Townsend (1976) has brought into evidence the existence of a region where the shear stress varies linearly while the velocity varies as the square root of the wall distance, which we shall refer to as the square root region. This region lies above the logarithmic region, i.e. where $p^+y^+ \gg 1$. The variables in the square root region are written in dimensionless form as $\hat{u} = \frac{u}{u_\tau p^+}$; $\hat{y} = \frac{y u_\tau}{\nu p^+}$; $\hat{k} = \frac{k}{u_\tau^2 p^+}$; $\hat{\varepsilon} = \frac{\nu\varepsilon}{u_\tau^4 p^{+2}}$; $\hat{\nu}_t = \frac{\nu_t}{\nu p^{+2}}$ while equation (7) reduces to $\hat{\nu}_t \frac{\partial \hat{u}}{\partial \hat{y}} = \hat{y}$.

Power law solutions are looked for. The variables are expanded as

$$\hat{u} = A_u \hat{y}^p \quad \hat{k} = A_k \hat{y}^q \quad \hat{\varepsilon} = A_\varepsilon \hat{y}^r$$

Substituting these expressions into equations (3, 4, 5), the balance of the exponents is fulfilled when $p = \frac{1}{2}, q = 1$ and $r = \frac{1}{2}$. Any model which is dimensionally consistent, as the classical models and the proposed model, satisfies the power law relations in the square root region.

The solution reads

$$A_k^2 = \frac{Q - C_{\varepsilon_1} R}{C_\mu (Q - C_{\varepsilon_2} R)} \quad A_u^2 = A_k \frac{Q - C_{\varepsilon_1} R}{C_{\varepsilon_2} - C_{\varepsilon_1}} \tag{9}$$
$$Q = \frac{\alpha}{2} + \beta - 2\gamma + \frac{1}{\sigma_{\varepsilon\varepsilon}} + \frac{3}{\sigma_{\varepsilon k}} \quad R = \frac{3}{\sigma_{kk}} + \frac{3}{2\sigma_{k\varepsilon}}$$

Most models cannot find such a solution since they give negative values for A_u and A_k. This point has been arrived at independently by Rao and Hassan (1998) for the $k-\omega$ model. Moreover, expected values are known: $A_u = \frac{2}{\kappa}$ and $A_k \sim \frac{1}{2a_1}$

Behaviour at a laminar/turbulent interface

At the edge of a turbulent region, the model must predict smooth behaviour of the mean flow as well as of the transported quantities. Otherwise, the prediction may be too sensitive to free-stream values. The analysis is similar to the one proposed by Cazalbou et al. (1994).

The analysis is performed for a thin layer (boundary layer, jet wake...). A self-similar form near the boundary ($y = \delta$) is assumed for the longitudinal velocity profile as

$$u = U_{\text{ext}} - u_\tau F'(\eta) \qquad \eta = \frac{y}{\sqrt{C_\mu}\delta} \tag{10}$$

where prime denotes differentiation with respect to η The vertical velocity profile can be deduced with the help of the continuity equation.

Similarly, self similar solutions are assumed for the turbulent kinetic energy and its dissipation rate

$$k = u_\tau^2 K(\eta) \qquad \varepsilon = \frac{u_\tau^3}{\delta} E(\eta) \tag{11}$$

The behaviour near the turbulent region edge is studied. Therefore, the following change of variable is used: $\lambda = \eta_{\text{ext}} - \eta$ and solutions are sought for as

$$F(\eta) = F_{\text{ext}} - A\lambda^a \quad K(\eta) = B\lambda^b \quad E(\eta) = C\lambda^c$$

where A, B and C must be strictly positive and a smooth behaviour is obtained only when

$$a \geq 2 \quad b \geq 1 \quad c \geq 1 \tag{12}$$

so that all quantities and their derivatives tend towards zero at the interface.

Leading order terms in the momentum equation give the first equality

$$2b - c - 1 = 0 \tag{13}$$

The balance of the turbulent kinetic energy and dissipation rate transport equations are chosen so that the production is negligible compared to the advection and the diffusion. This leads to the constraint

$$b < 2(a-1) \tag{14}$$

The balance of the advection and diffusion terms in both transport equations yield the following relations

$$b = [1 + (a-1)\sigma_{k\varepsilon}] \frac{\sigma_{kk}}{\sigma_{k\varepsilon} + 2\sigma_{kk}} \tag{15}$$
$$\gamma b^2 = c\left[(a-1) - b\left(\frac{1}{\sigma_{\varepsilon k}} + \beta\right) - c\left(\frac{1}{\sigma_{\varepsilon\varepsilon}} + \alpha\right)\right] \tag{16}$$

The exponents a, b and c can be deduced from equations (13, 15, 16). Because of equation (16) which is quadratic in b when $\gamma \neq 0$, second order equations are obtained for each coefficient. All the above equations and constraints (13-16) have been obtained assuming that a, b and c are positive. Negative values for a, b and c are not consistent with the prescribed edge values for the velocity, the turbulent kinetic energy and its dissipation rate. Therefore, to obtain the correct model behaviour, there must exist at least one positive value for a, b and c and all positive solutions must satisfy the constraints (12, 14).

It has been checked that the $k-\varepsilon$ and $k-\varphi$ models satisfy the constraints, while the $k-\omega$ model is known not to.

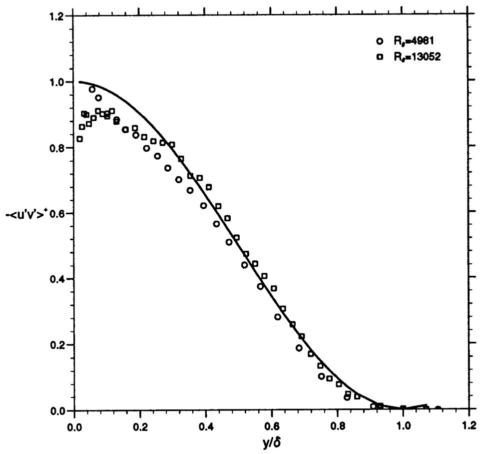

Figure 3. Comparison of equation (18) with the zero pressure gradient boundary layer experiment of Smith and Smits

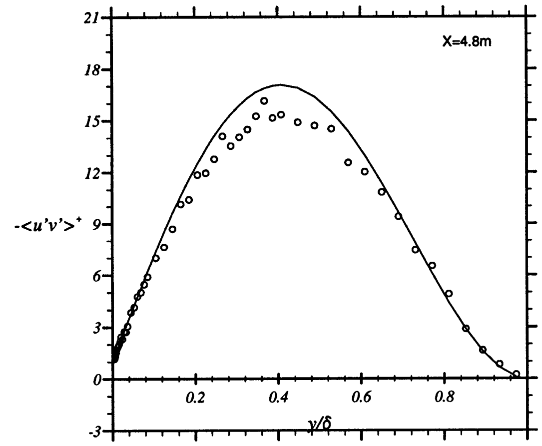

Figure 4. Comparison of equation (18) with the adverse pressure gradient boundary layer experiment of Skåre and Krogstad

Wake region

A way to optimize the prediction of the wake region has also been looked for. Self-similar solutions of the form presented above (eq. 10, 11) are used. Following Coles (1956), the velocity profile can be expressed as

$$\frac{U_{\text{ext}}-u}{u_\tau} = -\frac{1}{\kappa}\ln\eta + \frac{\Pi}{\kappa}2\cos^2\left(\frac{\pi}{2}\eta\right) \qquad \eta = \frac{y}{\delta} \tag{17}$$

where the wake strength Π is related to boundary layer integral thicknesses through

$$1+\Pi = \kappa\frac{\delta_1}{\delta}\sqrt{\frac{2}{C_f}} = \kappa F(1)$$

Assuming self-similarity, a form for the Reynolds stress profile $-<u'v'>/u_\tau^2(\eta)$ can be deduced from the momentum equation. The rather complex formulation is well approximated by

$$\frac{-<u'v'>}{u_\tau^2} = \left(1+\beta^*\frac{U_{\text{ext}}}{u_\tau}\eta\right)\cos^2\left(\frac{\pi}{2}\eta\right) \qquad \beta^* = -\frac{\delta}{u_\tau}\frac{dU_{\text{ext}}}{dx} \tag{18}$$

This simpler form is in good agreement with experiments, as shown in figures 3 and 4.

The model equations (3, 4, 5) can also be written in self-similar form as

$$-2\beta^*K - \left(\frac{1}{F_1}+2\beta^*\right)\eta K' = NF''^2 - E + \frac{d}{d\eta}\left(\frac{N}{\sigma_{kk}}K' + \frac{N}{\sigma_{k\varepsilon}}\frac{K}{E}E'\right) \tag{19}$$

$$-\left(\frac{1}{F_1}+6\beta^*\right)E - \left(\frac{1}{F_1}+2\beta^*\right)\eta E' = \left(C_{\varepsilon_1}NF''^2 - C_{\varepsilon_2}E\right)\frac{E}{K} + N\left(\alpha\frac{E'E'}{E} + \beta\frac{K'E'}{K} + \gamma\frac{EK'K'}{K^2}\right) + \frac{d}{d\eta}\left(\frac{N}{\sigma_{k\varepsilon}}\frac{E}{K}K' + \frac{N}{\sigma_{\varepsilon\varepsilon}}E'\right) \tag{20}$$

$$N = C_\mu\frac{K^2}{E} \tag{21}$$

where N is the dimensionless eddy viscosity $\nu_t/(u_\tau\delta)$ which can be deduced from (17, 18) and $F_1 = F(1)$. The above set of equations can be used to optimize the constants with reference to self-similar solutions.

A simple way to deal with the above system of equations is to use the Bradshaw assumption $-<u'v'>= 2a_1k$ which is in good agreement with experiment as shown in figure 5. Thus, K is known analytically and E can be deduced from the eddy viscosity expression. As only approximate expressions for F', K, E and N are obtained, the model coefficients can thus be estimated by minimizing the error on the balances of equations (19, 20) for a given range (say $0.2 \leq \eta \leq 0.8$) where all the assumptions hold.

APPLICATION TO MODEL DERIVATION

The above physical constraints are general. Their derivation has been restricted to the proposed model, with an eddy viscosity assumption. The key problem comes from the eddy viscosity assumption which gives, for a two-dimensional boundary layer, $-<u'v'>= 2a_1\sqrt{\frac{P_k}{\varepsilon}}k$. In the boundary layer, the regions where $P_k = \varepsilon$ and where Bradshaw's assumption is verified are different; the eddy viscos-

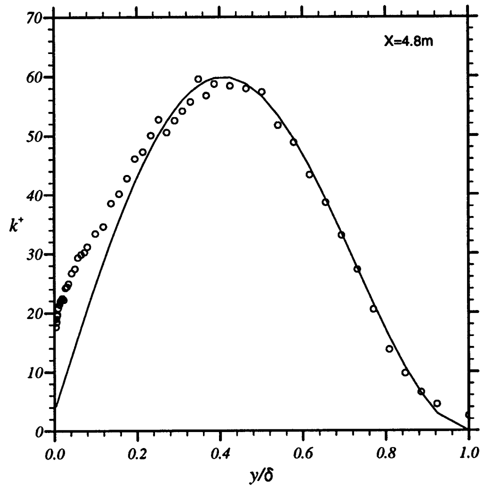

Figure 5. Comparison of the turbulent kinetic energy profile deduced from equation (18) and the Bradshaw's assumption with boundary layer experiment of Skåre and Krogstad

ity assumption is thus not fully consistent with the Bradshaw's assumption.

The main goal for the model is to correctly predict the mean velocity profile, and thus the Reynolds stress $-<u'v'>$ and the eddy viscosity but not necessarily the turbulent kinetic energy. Hence, in order to get good predictions with an eddy viscosity assumption, k should not strictly be the turbulent kinetic energy but just the velocity scale used to compute the eddy viscosity. Thus, the self-similar form proposed for k in the wake region should not be used. Similarly, A_k cannot be equal to $1/2a_1$ in the square root region.

Therefore, an eddy viscosity model can just been asked to satisfy the isotropic turbulence decay, the slope of the logarithmic region for zero or moderate pressure gradients, the slope A_u in the square root region (but not A_k) and the behaviour at an interface. No simple analytical constraint can be applied in the wake region, numerical optimization has to be used. However, there is still a family of sets of constants which satisfies all the analytical constraints. Such models are presently being tested.

When the eddy viscosity assumption is relaxed, either using an algebraic stress formulation or a full Reynolds stress approach, consistency with the Bradshaw hypothesis can be expected. However, because of the change of the constitutive relation, the mathematical form of the constraints is somehow different. Moreover, the diffusion model can be altered to take advantage of the knowledge of the Reynolds stress tensor anisotropy.

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