THE HOMOGENIZATION OF NAVIER-STOKES EQUATIONS AND A NOVEL CLOSURE FOR FULLY DEVELOPED TURBULENCE [1]

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ABSTRACT

In this paper we reformulate a homogenization of Navier-Stokes equations in generalized form and then according to Kolmogorov's scaling theory make an analysis to get a novel closure for fully developed turbulence. Our research in physics is essentially different from those available in the literature, although in some mathematical method it follows Chacon at al (1994). We also explore a novel method to calculate the closure terms in homogenized quations for fully developed turbulence.

INTRODUCTION

In modern mathematics, a theory has been developed called homogenization which deals with partial differential equations having rapidly oscilating coefficients (Jikov et al. 1994). In that huge and fascinating field of homogenization multiscale methods based on asymptotic tools are extensively used. Typically, the questions which can be addressed by them are of the following form: a 'small-scale' structure with some form of translation invariance (periodicity, quasi-periodicity, random homogeneity) is prescribed. Thus, homogenization can handle a variety of different important physical and engineering problems.

Unfortunately, up to now, a little effort has been made to bridge gap between these different fields of research. At this moment, we have not yet taken advantage of the full power of the mathematical tools from homogenization theory. Many known theoretical results have not yet been used for numerical simulation of practical problems in research and industry. There seems to be a very good chance that at least some of these problems may be successfully solved in the near future (Hornung 1997). Now we make an effort to explore the homogenization theory as a new tool for studying the fluid turbulence. In this paper we first present a generalized formulation of homogenization of Navier-Stokes equations with two different length scales and then based on Kolmogorov's scale theory give their implements within the inertial range placing the analysis as a study of the fully developed turbulence leading a novel closure system, and finally, explore a method to calculate the closure terms in homogenized equations for fully developed turbulence.

GENERALIZED FORMULATION FOR HOMOGENIZATION OF NAVIER-STOKES EQUATIONS

In the classic homogenization theory one assumes that every property of the field is of the form $f(x,y)$ where $y = x/\epsilon$. Here $x = (x_1, \cdots, x_N)$ is the N-dimensional position vector of a point in Cartesian coordinates and $y = (y_1, \cdots, y_N)$ is the vector of stretched coordinates. One shall look for each unknown field quantity $u^{\epsilon}(x)$ in the form of a doublescale asymptotic expansion:

$$u^{\epsilon}(x) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \cdots$$

The two-scale process introduced in the partial differential equations of the problem produces equations in x

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and y variables. Generally speaking, equations in y are solvable if the microstructure is, in some sense, periodic ; and terms $u_i(x,y)$ in the postulated asymptotic expansions are periodic in the y variable with the same period as that of the structure. Moreover this leads to a rigorous deductive procedure for obtaining the macroscopic equations (in x) for the global behavior of the field. In homogenization it is usual to consider first x and y as independent variables and to replace next y by x/ϵ . Applied to a function $u^\epsilon(x,x/\epsilon)$, the operator ∇ becomes

$$\nabla u^\epsilon(x,x/\epsilon)=\nabla_x u^\epsilon(x,y)+\frac{1}{\epsilon}\nabla_y u^\epsilon(x,y).$$

The solution is expanded in powers of ϵ and substituted into the equations with decomposed derivatives . Comparing the coefficients of the different ϵ powers in this equation, we can obtain the equations for the various orders in ϵ. The large-scale equation always emerges as a solvability condition to some order in ϵ . Integrating these identities over Y and proceeding further we can get the homogenized limit of the studied equation. The values of the coefficients in the large-scale equation are obtained in terms of the solution of the lower order equations (Hornung 1997). In this paper we generalize the homogenization to the non-linear unsteady Navier-Stokes equations. In a generalized homogenization we assume that the scalar- or vector-valued field depends not only on the slow variables x and t, but also on the fast variables $y=\frac{a}{\epsilon}$ and $\tau=\frac{t}{\epsilon^s}$, where $a=(a_1,\cdots,a_N)=a(x,t)$ is a N-dimensional position vector of properly transformed coordinates that allows the considered field to be smooth and almost-periodic in the y-τ variables, choice of the coordinates $a(x,t)$ and the value of the exponent s depends mostly on the symmetries of the problem. Since the solutions depend on x and t both through the fast variables and through the slow variables, space-and time-derivatives must be decomposed as follows:

$$\begin{aligned}\frac{\partial}{\partial t}&=\partial_t+\epsilon^{-s}\partial_\tau+\epsilon^{-1}a_t\cdot\nabla_y\,,\\ \nabla&=\nabla_x+\epsilon^{-1}\nabla a\cdot\nabla_y\,.\end{aligned}$$

The solution is expanded in powers of ϵ^{β} and substituted into the equations (with decomposed derivatives) where the value of the exponent β with the previous s in τ can not be an integer. The main new difficulty for the nonlinear equations is to find the order in ϵ^{β} of the leading term of the solution: this is often done by a dominant balance argument in the large-scale equation (the form of which can usually be guessed by symmetry arguments). The final equation, which emerges again as a solvability condition, is still nonlinear. In homogenization theory the multiple scales expansion is a simple tool to obtain equations for the means when there are periodic or quarsi-periodic oscillations. In turbulence model the assumptions of quasi-periodicity can be made on the scales in the velocity and pressure fields with respect to the variables y and τ . In this paper we consider the Navier-Stokes equations with a small parameter ϵ in the viscosity and initial velocity fluctuation :

$$\begin{aligned}&\frac{\partial}{\partial t}u^\epsilon+(u^\epsilon\cdot\nabla)u^\epsilon-\frac{1}{\mathrm{Re}}\Delta u^\epsilon+\nabla p^\epsilon\\ =&\ 0\,,\qquad \text{in } \mathbb{R}^{\mathrm{N}}\times(0,T]\end{aligned}\tag{1}$$

$$\nabla\cdot u^\epsilon=0\,,\qquad \text{in } \mathbb{R}^{\mathrm{N}}\times(0,T]\tag{2}$$

$$u^\epsilon(x,0)=u_{(0)}^{(0)}(x)+\epsilon^\beta u_{(1)}^{\prime(0)}(x,\frac{x}{\epsilon})\quad \text{in } \mathbb{R}^{\mathrm{N}}.\tag{3}$$

where $u_{(1)}^{\prime(0)}(x,y)$ is almost periodic in $y=\frac{x}{\epsilon}$ and has zero y-means, e.i., $\langle u_{(1)}^{\prime(0)}(x,y)\rangle=0$, $\frac{1}{\mathrm{Re}}$ can be represented as $\frac{1}{\mathrm{Re}}=\epsilon^\alpha\nu_0$. The values of the exponents α and β depend on the considerded problems of turbulence.

We assume that the velocity u^ϵ and the pressure p^ϵ fields admit asymptotic expansions of the form:

$$\begin{aligned}&u^\epsilon(x,t)\\ =&\ u_{(0)}(x,t)+\epsilon^\beta u'_{(1)}(x,t\,;y,\tau)+\epsilon^{2\beta}u'_{(2)}(x,t\,;y,\tau)\\ &+\epsilon^{3\beta}u'_{(3)}(x,t\,;y,\tau)+\cdots\,,\end{aligned}\tag{4}$$

$$\begin{aligned}&p^\epsilon(x,t)\\ =&\ p_{(0)}(x,t)+\epsilon^\beta p'_{(1)}(x,t\,;y,\tau)+\epsilon^{2\beta}p'_{(2)}(x,t\,;y,\tau)\\ &+\epsilon^{3\beta}p'_{(3)}(x,t\,;y,\tau)+\cdots\,.\end{aligned}\tag{5}$$

and all functions $u'_{(k)}(x,t;y,\tau)$, $p'_{(k)}(x,t;y,\tau)$ are smooth and almost-periodic in the y-τ variables.

To obtain the formal equations for the terms of expansions, we derive the multiscale expansion of expanded Euler operator for (u^ϵ,p^ϵ) :

$$E(u^\epsilon,p^\epsilon)=\nabla a^T\cdot\left[\frac{\partial u^\epsilon}{\partial t}+\nabla\cdot(u^\epsilon\otimes u^\epsilon)+\nabla p^\epsilon\right],\tag{6}$$

and multiscale expansion of expanded Laplacian operator for u^ϵ :

$$Lu^\epsilon=\nabla a^T\cdot(\Delta u^\epsilon)=\nabla a^T\cdot(\nabla\cdot\nabla u^\epsilon)\,,\tag{7}$$

and multiscale expansion of Divergence operator for u^ϵ :

$$Du^\epsilon=\nabla\cdot u^\epsilon\,.\tag{8}$$

For scales consistency, we should have $\beta=1-s$. Thus, substuting the expansions (4) and (5) with the decomposed derivatives into the operators (6), (7) and (8), we can express the Navier-Stokes equations in the form of multiscale expansions:

$$E(u^\epsilon,p^\epsilon)-\epsilon^\alpha\nu_0 Lu^\epsilon=0\,,\tag{9}$$

$$Du^{\epsilon} = 0. \tag{10}$$

Substituting the expansions of the Euler, Laplacian and Divergence operators (6), (7), (8) into the equations (9) and (10), pulling their terms for the various orders in ϵ^{β} to be equal to zero, we can get the formal equations for the terms of expansions (4) and (5). For example, if we choose $\beta = \frac{1}{3}$ and $\frac{\partial a}{\partial t} + u \cdot \nabla a = 0$, introduce variables $\tilde{u}'_{(i)} = \nabla a^T \cdot u'_{(i)}$, those equations can be represented as follows:

$$\nabla_y \cdot \left[p'_{(1)} \left(\nabla a^T \cdot \nabla a\right)\right] = \nu_0 L_{(i)}, \tag{11}$$

$$\nabla_y \cdot \tilde{u}'_{(1)} = 0, \tag{12}$$

$$\begin{aligned} &\frac{\partial \tilde{u}'_{(1)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(1)}\right) \\ &+\nabla_y \cdot \left[p'_{(2)} \left(\nabla a^T \cdot \nabla a\right)\right] \\ = \;& \nu_0 L_{(i+1)}, \end{aligned} \tag{13}$$

$$\nabla_y \cdot \tilde{u}'_{(2)} = 0, \tag{14}$$

$$\begin{aligned} &\frac{\partial \tilde{u}'_{(2)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(2)}\right) \\ &+\nabla_y \cdot \left(\tilde{u}'_{(2)} \otimes \tilde{u}'_{(1)}\right) + \nabla_y \cdot \left[p'_{(3)} \left(\nabla a^T \cdot \nabla a\right)\right] \\ &+\nabla a^T \cdot \left[\frac{\partial u_{(0)}}{\partial t} + \nabla_x \cdot \left(u_{(0)} \otimes u_{(0)}\right)\right. \\ &\left. + \nabla_x p_{(0)}\right] \\ = \;& = \nu_0 L_{(i+2)}, \end{aligned} \tag{15}$$

$$\nabla_y \cdot \tilde{u}'_{(3)} = -\nabla_x \cdot u_{(0)}\,, \tag{16}$$

$$\begin{aligned} &\frac{\partial \tilde{u}'_{(3)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(3)}\right) \\ &+\nabla_y \cdot \left(\tilde{u}'_{(2)} \otimes \tilde{u}'_{(2)}\right) + \nabla_y \cdot \left(\tilde{u}'_{(3)} \otimes \tilde{u}'_{(1)}\right) \\ &+\nabla_y \cdot \left[p'_{(4)} \left(\nabla a^T \cdot \nabla a\right)\right] \\ &+\nabla a^T \cdot \left[\frac{\partial u'_{(1)}}{\partial t} + \nabla_x \cdot \left(u_{(0)} \otimes u'_{(1)}\right)\right. \\ &\left.+\nabla_x \cdot \left(u'_{(1)} \otimes u_{(0)}\right) + \nabla_x p'_{(1)}\right] \\ = \;& \nu_0 L_{(i+3)}, \end{aligned} \tag{17}$$

$$\nabla_y \cdot \tilde{u}'_{(4)} = -\nabla_x \cdot u'_{(1)}, \tag{18}$$

$$\begin{aligned} &\frac{\partial \tilde{u}'_{(4)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(4)}\right) \\ &+\nabla_y \cdot \left(\tilde{u}'_{(2)} \otimes \tilde{u}'_{(3)}\right) + \nabla_y \cdot \left(\tilde{u}'_{(3)} \otimes \tilde{u}'_{(2)}\right) \\ &+\nabla_y \cdot \left(\tilde{u}'_{(4)} \otimes \tilde{u}'_{(1)}\right) \\ &+\nabla_y \cdot \left[p'_{(5)} \left(\nabla a^T \cdot \nabla a\right)\right] \\ &+\nabla a^T \cdot \left[\frac{\partial u'_{(2)}}{\partial t} + \nabla_x \cdot \left(u_{(0)} \otimes u'_{(2)}\right)\right. \\ &+\nabla_x \cdot \left(u'_{(2)} \otimes u_{(0)}\right) \\ &\left.+\nabla_x \cdot \left(u'_{(1)} \otimes u'_{(1)}\right) + \nabla_x p'_{(2)}\right] \\ = \;& \nu_0 L_{(i+4)}, \end{aligned} \tag{19}$$

$$\nabla_y \cdot \tilde{u}'_{(5)} = -\nabla_x \cdot u'_{(2)}, \tag{20}$$

$$\begin{aligned} &\frac{\partial \tilde{u}'_{(5)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(5)}\right) \\ &+\nabla_y \cdot \left(\tilde{u}'_{(2)} \otimes \tilde{u}'_{(4)}\right) + \nabla_y \cdot \left(\tilde{u}'_{(3)} \otimes \tilde{u}'_{(3)}\right) \\ &+\nabla_y \cdot \left(\tilde{u}'_{(4)} \otimes \tilde{u}'_{(2)}\right) + \nabla_y \cdot \left(\tilde{u}'_{(5)} \otimes \tilde{u}'_{(1)}\right) \\ &+\nabla_y \cdot \left[p'_{(6)} \left(\nabla a^T \cdot \nabla a\right)\right] \\ &+\nabla a^T \cdot \left[\frac{\partial u'_{(3)}}{\partial t} + \nabla_x \cdot \left(u_{(0)} \otimes u'_{(3)}\right)\right. \\ &+\nabla_x \cdot \left(u'_{(3)} \otimes u_{(0)}\right) + \nabla_x \cdot \left(u'_{(1)} \otimes u'_{(2)}\right) \\ &\left.+\nabla_x \cdot \left(u'_{(2)} \otimes u'_{(1)}\right) + \nabla_x p'_{(3)}\right] \\ = \;& \nu_0 L_{(i+5)}, \end{aligned} \tag{21}$$

$$\cdots\cdots$$

Here the index i of the right side of (11), (13), (15),(15), (17), (19) depends on the choice of the values of the exponents α for $\frac{1}{\mathrm{Re}} = \epsilon^{\alpha}\nu_0$: $i = 1, 2, 3, 4, 5, 6$ correspond to $\alpha = 2, \frac{5}{3}, \frac{4}{3}, 1, \frac{2}{3}, \frac{1}{3}$ respectively; and

$$L_{(1)} = 0, \qquad L_{(2)} = 0, \quad L_{(3)} = 0,$$

$$L_{(4)} = \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'_{(1)}\right],$$

$$L_{(5)} = \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'_{(2)}\right],$$

$$L_{(6)} = \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'_{(3)}\right].$$

$$\cdots\cdots$$

ANALYSIS FOR FULLY DEVELOPED TURBULENCE

One of the problem of our analysis for turbulent flows is that we may approximate u by a function with multiple scales, but there are no obvious period. But in asymptotic sense we can approximate u by a quasi-periodic function, for which the space, time and space-time means are defined by

$$\langle u\rangle = \lim_{r\to\infty} \frac{1}{B(x,r)} \int_{B(x,r)} u(x,t\,;\,y,\tau)dy,$$

$$\overline{u} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(x,t\,;\,y,\tau)d\tau,$$

$$\overline{\langle u\rangle} = \lim_{\substack{T\to\infty\\ r\to\infty}} \frac{\int_{-T}^{T}\int_{B(x,r)} u(x,t\,;y,\tau)\,dy\,d\tau}{2TB(x,r)}$$

It is easy to prove that if u and $[\cdot\cdots\cdot\cdot]$ are almost-periodic and smooth enough, then the following identities hold:

$$\langle \nabla y\cdot[\cdot\cdots\cdot\cdot]\rangle = 0, \qquad (22)$$

$$\overline{\frac{\partial u}{\partial\tau}} = 0, \qquad (23)$$

$$\overline{\langle \frac{\partial u}{\partial\tau} + \nabla y\cdot[\cdot\cdots\cdot\cdot]\rangle} = 0. \qquad (24)$$

We use the Kolmogorov's scaling theory for fully developed turbulence, assuming that ℓ is the scale under consideration, v_ℓ is the typical value of the velocity associated to scales $\backsim \ell$ and the 'eddy' turnover time (circulation time) associated with the scale ℓ : $t_\ell \backsim \frac{\ell}{v_\ell}$ is the typical time for a structure of size $\backsim \ell$ to undergo a significant distortion due to the relative motion of its components, then the velocity field v_ℓ is scale-invariant of exponent $\beta = 1/3$: $v_\ell \backsim \varepsilon^{1/3}\ell^{1/3}$ and the eddy turnover time is $t_\ell \backsim \varepsilon^{-1/3}\ell^{2/3}$ where ε is the finite mean energy dissipation rate that is independent on ℓ in the inertial range (Frisch 1995). So, the right choice in the multiscale expansions for an analysis of fully developed turbulence is $\epsilon = \ell/L$, $\beta = \frac{1}{3}$, $s = 1-\beta = \frac{2}{3}$. The coordinate function $a = a(x,t)$ is chosen as Lagrangian coordinate: $\frac{\partial a}{\partial t} + u\cdot\nabla a = 0$, $a(x,0) = x$ that is boundary-fitted and allows the velocity and pressure fluctuations to be almost-periodic in the $y = \frac{a}{\epsilon}$ and $\tau = \frac{t}{\epsilon^{2/3}}$ variables.

However, we have many choices of the values of the exponents for $\frac{1}{\mathrm{Re}} = \epsilon^{\alpha}\nu_0$:

(1) if $\alpha = \frac{1}{3}, \frac{2}{3}$, then

$$\lim_{\epsilon\to 0} \epsilon\,\mathrm{Re} \backsim O\left(\epsilon^{1-\alpha}\right) \to 0,$$

as $\ell \to 0$, $\nu > 0$. It is impossible physically .

(2) if $\alpha = 1$, then

$$\lim_{\epsilon\to 0} \epsilon\,\mathrm{Re} \backsim O\ (1)\ ,$$

as $\ell \to 0\,, \nu \to 0$. In this case the system of equations can not be closure: the equations for the lower order terms contain the higher order terms.

(3) if $\alpha = \frac{4}{3}, \frac{5}{3}, 2$, then

$$\lim_{\epsilon\to 0} \epsilon\,\mathrm{Re} \backsim O\left(\epsilon^{1-\alpha}\right) \to \infty\,.$$

the limit $\nu \to 0$ is taken before the limit $\ell \to 0$.

So the right choice is $\alpha \gtrsim 4/3$. The linear scale of the motion in which viscous dissipation occur: $\lambda = \nu^{3/4}\langle\varepsilon\rangle^{-\frac{1}{4}}$ (Frish 1995). In this paper we take $\alpha = 4/3$, i.e., $\frac{1}{\mathrm{Re}} = \epsilon^{4/3}\nu_0$, then the following solvability conditions hold:

$$\begin{aligned} &\overline{\langle \frac{\partial u_0}{\partial t} + u_{(0)}\cdot\nabla_x u_{(0)} + \nabla_x p_{(0)}\rangle} \\ = &\ \frac{\partial u_0}{\partial t} + u_{(0)}\cdot\nabla_x u_{(0)} + \nabla_x p_{(0)} = 0\ , \end{aligned} \qquad (25)$$

$$\overline{\langle \nabla_x\cdot u_{(0)}\rangle} = \nabla_x\cdot u_{(0)} = 0, \qquad (26)$$

$$\begin{aligned} &\overline{\langle \frac{\partial u'_{(1)}}{\partial t} + u_{(0)}\cdot\nabla_x u'_{(1)}} \\ &\overline{+u'_{(1)}\cdot\nabla_x u_{(0)} + \nabla_x p'_{(1)}\rangle} \\ = &\ \frac{\partial\overline{\langle u'_{(1)}\rangle}}{\partial t} + u_{(0)}\cdot\nabla_x\overline{\langle u'_{(1)}\rangle} && (27) \\ &+\overline{\langle u'_{(1)}\rangle}\cdot\nabla_x u_{(0)} + \nabla_x\overline{\langle p'_{(1)}\rangle} && (28) \\ = &\ 0, && (29) \end{aligned}$$

$$\overline{\langle \nabla_x\cdot u'_{(1)}\rangle} = \nabla_x\cdot\overline{\langle u'_{(1)}\rangle} = 0, \qquad (30)$$

$$\begin{aligned} &\overline{\langle \frac{\partial u'_{(2)}}{\partial t} + u_{(0)}\cdot\nabla_x u'_{(2)} + u'_{(2)}\cdot\nabla_x u_{(0)}} \\ &\overline{+\nabla_x\cdot\left(u'_{(1)}\otimes u'_{(1)}\right) + \nabla_x p'_{(2)}\rangle} \\ = &\ \frac{\partial\overline{\langle u'_{(2)}\rangle}}{\partial t} + u_{(0)}\cdot\nabla_x\overline{\langle u'_{(2)}\rangle} + \overline{\langle u'_{(2)}\rangle}\cdot\nabla_x u_{(0)} \\ &+\nabla_x\cdot\overline{\langle\left(u'_{(1)}\otimes u'_{(1)}\right)\rangle} + \nabla_x\overline{\langle p'_{(2)}\rangle} && (31) \\ = &\ 0, && (32) \end{aligned}$$

$$\overline{\langle \nabla_x\cdot u'_{(2)}\rangle} = \nabla_x\cdot\overline{\langle u'_{(2)}\rangle} = 0, \qquad (33)$$

$$
\begin{aligned}
& \overline{\langle \frac{\partial u'_{(3)}}{\partial t} + u_{(0)} \cdot \nabla_x u'_{(3)} + u'_{(3)} \cdot \nabla_x u_{(0)}} \\
& \overline{+\nabla_x \cdot \left(u'_{(1)} \otimes u'_{(2)}\right) + \nabla_x \cdot \left(u'_{(2)} \otimes u'_{(1)}\right)} \\
& \overline{+\nabla_x p'_{(3)} \rangle} \\
= & \frac{\partial \overline{\langle u'_{(3)} \rangle}}{\partial t} + u_{(0)} \cdot \nabla_x \overline{\langle u'_{(3)} \rangle} + \overline{\langle u'_{(3)} \rangle} \cdot \nabla_x u_{(0)} \\
& + \nabla_x \cdot \overline{\langle \left(u'_{(1)} \otimes u'_{(2)}\right) \rangle} \\
& + \nabla_x \cdot \overline{\langle \left(u'_{(2)} \otimes u'_{(1)}\right) \rangle} + \nabla_x \overline{\langle p'_{(3)} \rangle} \\
= & 0,
\end{aligned} \tag{34}
$$

$$
\overline{\langle \nabla_x \cdot u'_{(3)} \rangle} = \nabla_x \cdot \overline{\langle u'_{(3)} \rangle} = 0, \tag{35}
$$

$$\cdots\cdots.$$

When all the solvability conditions are satisfied, we have equations:

$$
\begin{aligned}
& \frac{\partial \tilde{u}'_{(1)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(1)}\right) \\
& + \nabla_y \cdot \left[p'_{(2)} \left(\nabla a^T \cdot \nabla a\right)\right] \\
= & \nu_0 \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'_{(1)}\right],
\end{aligned} \tag{36}
$$

$$
\nabla_y \cdot \tilde{u}'_{(1)} = 0, \tag{37}
$$

$$
\begin{aligned}
& \frac{\partial \tilde{u}'_{(2)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(2)}\right) \\
& + \nabla_y \cdot \left(\tilde{u}'_{(2)} \otimes \tilde{u}'_{(1)}\right) \\
& + \nabla_y \cdot \left[p'_{(3)} \left(\nabla a^T \cdot \nabla a\right)\right] \\
= & \nu_0 \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'_{(2)}\right],
\end{aligned} \tag{38}
$$

$$
\nabla_y \cdot \tilde{u}'_{(2)} = 0, \tag{39}
$$

$$
\begin{aligned}
& \frac{\partial \tilde{u}'_{(3)}}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}'_{(1)} \otimes \tilde{u}'_{(3)}\right) \\
& + \nabla_y \cdot \left(\tilde{u}'_{(2)} \otimes \tilde{u}'_{(2)}\right) + \nabla_y \cdot \left(\tilde{u}'_{(3)} \otimes \tilde{u}'_{(1)}\right) \\
& + \nabla_y \cdot \left[p'_{(4)} \left(\nabla a^T \cdot \nabla a\right)\right] \\
= & \nu_0 \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'_{(3)}\right],
\end{aligned} \tag{40}
$$

$$
\nabla_y \cdot \tilde{u}'_{(3)} = 0, \tag{41}
$$

$$\cdots\cdots.$$

where $\tilde{u}'_{(k)} = \nabla a^T \cdot u'_{(k)}$.

This problem admits an unique almost-periodic solution for $\left(u'_{(k)}, p'_{(k)}\right)$, which depends smoothly on y, τ and the tensor $\nabla a^T \cdot \nabla a$. So, the averaged tensors $\overline{\langle u'_{(i)} \otimes u'_{(j)} \rangle}$ are the functionals of $\nabla a^T \cdot \nabla a$, which will be shown in the next section.

From these solvability conditions one can get the averaged equations for $\left(u_{(0)}, p_{(0)}\right)$, $\left(\overline{\langle u'_{(1)} \rangle}, \overline{\langle p'_{(1)} \rangle}\right)$, $\left(\overline{\langle u'_{(2)} \rangle}, \overline{\langle p'_{(2)} \rangle}\right)$and $\left(\overline{\langle u'_{(3)} \rangle}, \overline{\langle p'_{(3)} \rangle}\right)$.

We can show that $\overline{\langle u'_{(1)} \rangle} = 0$, $\overline{\langle p'_{(1)} \rangle}$ =constant.

If $\left(\overline{\langle u'_{(2)} \rangle}, \overline{\langle p'_{(2)} \rangle}\right)$and $\left(\overline{\langle u'_{(3)} \rangle}, \overline{\langle p'_{(3)} \rangle}\right)$are kept in the result and $\overline{\langle u^{\epsilon} \rangle} = u_0 + \epsilon^{2/3} \overline{\langle u'_{(2)} \rangle} + \epsilon \overline{\langle u'_{(3)} \rangle}$ is denoted by u, $\overline{\langle p^{\epsilon} \rangle} = p_0 + \epsilon^{2/3} \overline{\langle p'_{(2)} \rangle} + \epsilon \overline{\langle p'_{(3)} \rangle}$ denoted by p, then we get a closure system approximating the fully developed turbulence with the accuracy of order ϵ:

$$
\frac{\partial a}{\partial t} + u \cdot \nabla a = 0, \qquad a(x, 0) = x, \tag{42}
$$

$$
\begin{aligned}
& \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p + \nabla \cdot \mathbf{R} \\
= & 0, \qquad u(x, 0) = u^{(0)},
\end{aligned} \tag{43}
$$

$$
\nabla \cdot u = 0. \tag{44}
$$

where

$$
\mathbf{R} = \epsilon^{2/3} \overline{\langle \left(u'_{(1)} + \epsilon^{1/3} u'_{(2)}\right) \otimes \left(u'_{(1)} + \epsilon^{1/3} u'_{(2)}\right) \rangle} \tag{45}
$$

CALCULATION OF CLOSURE TERMS IN AVERAGED EQUATIONS

Let

$$
u' = u'_{(1)} + \epsilon^{1/3} u'_{(2)}, \quad \tilde{u}' = \nabla a^T \cdot u'; \tag{46}
$$

and

$$
p' = p'_{(2)} + \epsilon^{1/3} p'_{(3)}, \tag{47}
$$

we can combinate the equations (36) - (39) with the accuracy of considered problem into the equations:

$$
\begin{aligned}
& \frac{\partial \tilde{u}'}{\partial \tau} + \nabla_y \cdot \left(\tilde{u}' \otimes \tilde{u}'\right) \\
& + \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) p'\right] \\
= & \nu_0 \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \tilde{u}'\right],
\end{aligned} \tag{48}
$$

$$
\nabla_y \cdot \tilde{u}' = 0. \tag{49}
$$

We define m' as a vector field equivalent to u' up to a gradient of φ', similar to a gauge transformation of Yang-Mills kind in electromagnetics:

$$
m' = u' - \nabla a \cdot \nabla_y \varphi' \tag{50}
$$

and also

$$\widetilde{m}' = \widetilde{u}' - \nabla_y \left(\nabla a^T \cdot \nabla a\right) \varphi' \tag{51}$$

where $\widetilde{m}' = \nabla a^T \cdot m'$.

According to above transformation one can lead a new formulation of the equations (48) and (49) in a gauge-invariant form (Wu, 1999):

$$\begin{aligned} & \frac{\partial \widetilde{m}'}{\partial \tau} + \nabla_y \cdot (\, \widetilde{u}' \otimes \, \widetilde{m}') \\ = & \; - \left[\nabla a^T \cdot \nabla a \cdot \nabla_y u'\right] \cdot m' \\ & + \nu_0 \nabla_y \cdot \left[\nabla a^T \cdot \nabla a \cdot \nabla_y \widetilde{m}'\right] , \end{aligned} \tag{52}$$

$$\widetilde{u}' = \widetilde{m}' + \nabla a^T \cdot \nabla a \cdot \nabla_y \varphi' \tag{53}$$

$$\nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \varphi'\right] = -\nabla_y \cdot \widetilde{m}' \tag{54}$$

$$\begin{aligned} p' \; = & \\ & \frac{1}{2} \left|u'\right|^2 + \frac{\partial \varphi'}{\partial \tau} + \nabla_y \cdot (\, \widetilde{u}' \varphi') \\ & - \nu_0 \nabla_y \cdot \left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \varphi'\right] . \end{aligned} \tag{55}$$

Due to

$$\begin{aligned} & \overline{\langle \frac{\partial \varphi'}{\partial \tau} + \nabla_y \cdot \left(\widetilde{u}' \varphi' \right)} \\ & \overline{- \nu_0 \nabla_y \cdot \left(\nabla a^T \cdot \nabla a \cdot \nabla_y \varphi'\right)\rangle} \\ = & \; 0 , \end{aligned} \tag{56}$$

we have

$$\overline{\langle p' \rangle} = \overline{\langle \frac{1}{2} \left|u'\right|^2 \rangle} \tag{57}$$

In general, we can solve the above problem at least numerically and then determine all the closure terms:

$$\mathbf{R} = \epsilon^{2/3} \overline{\langle u' \otimes u' \rangle}, \tag{58}$$

$$\cdots\cdots$$

However, under some working hypothesis, it is possible to simplify notably the dependence of closure terms on the tensor $\nabla a^T \cdot \nabla a$.

From equation (48), we have

$$\begin{aligned} & \nabla_y \cdot \overline{\langle \widetilde{u}' \otimes \widetilde{u}' \rangle} \\ & + \nabla_y \cdot \overline{\left[p' \left(\nabla a^T \cdot \nabla a\right)\right]} \\ = & \; \nu_0 \nabla_y \cdot \overline{\left[\left(\nabla a^T \cdot \nabla a\right) \cdot \nabla_y \widetilde{u}'\right]} \end{aligned} \tag{59}$$

due to

$$\overline{\frac{\partial \, \widetilde{u}'}{\partial \, \tau}} = 0$$

For homogeneous turbulence, the following tensor relation holds:

$$\begin{aligned} & \overline{\langle \widetilde{u}' \otimes \widetilde{u}' \rangle} \\ & + \frac{1}{2} \left(\nabla a^T \cdot \nabla a\right) \overline{\langle u' \cdot u' \rangle} \\ = & \; \nu_0 \left(\nabla a^T \cdot \nabla a\right) \cdot \overline{\langle \nabla_y \widetilde{u}' \rangle} , \end{aligned} \tag{60}$$

and so we have:

$$\begin{aligned} & \overline{\langle \widetilde{u}' \otimes \widetilde{u}' \rangle} \\ = & \; \alpha_1 (x,t) \left(\nabla a^T \cdot \nabla a\right) \\ & + \alpha_2 (x,t) \left(\nabla a^T \cdot \nabla a\right) \cdot \left(\nabla a^T \cdot \nabla a\right) \end{aligned} \tag{61}$$

$$\begin{aligned} & \overline{\langle \nabla_y \widetilde{u}' \rangle} \\ = & \; \beta_1 (x,t) \, \mathbf{I} + \beta_2 (x,t) \left(\nabla a^T \cdot \nabla a\right) \end{aligned} \tag{62}$$

Thus, the Reynolds tensor is

$$\begin{aligned} R \; = & \; \epsilon^{2/3} \overline{\langle u' \otimes u' \rangle} = \epsilon^{2/3} \nabla a^{-T} . \overline{\langle \widetilde{u}' \otimes \widetilde{u}' \rangle} \cdot \nabla a^{-1} \\ = & \; \epsilon^{2/3} \left[\alpha_1 (x,t) \, \mathbf{I} + \alpha_2 (x,t) \left(\nabla a \cdot \nabla a^T\right)\right] . \end{aligned} \tag{63}$$

Concerning the details of calculation of the closure terms, they are discussed in the next paper (Wu, 1999) .

CONCLUSION

It has been shown that the homogenization of Navier-Stokes equations can be used to study the fluid turbulence. There is no doubt that this theory will yield good turbulence models in the future.

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