INVESTIGATION OF THE RENORMALIZATION GROUP USING NUMERICAL SIMULATIONS OF TURBULENCE

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ABSTRACT

We consider the possibility of obtaining an eddy-viscosity for large-eddy simulations by applying Renormalization-Group methods to the Navier-Stokes equation. We outline how the introduction of a conditional averaging procedure may allow us to perform such a calculation as an approximation, and look at how the results of direct numerical simulations can be used to test the assumptions made therein. Finally, we examine and discuss the results obtained by performing a large-eddy simulation using the resulting eddy-viscosity.

INTRODUCTION

Over a period of several years, our research in Edinburgh has been concerned with the application of renormalization methods to predict the energy spectrum of isotropic turbulence. In particular we have concentrated upon the development of **Renormalization Group (RG)** techniques (McComb et al., 1992a, McComb & Watt, 1992b) by which we may obtain predictions for various parameters of turbulence, for instance the Kolmogorov constant, and also obtain a subgrid model for spectral large-eddy simulations (LES).

The RG approach (see, for instance, Wilson 1975) has enjoyed a fair degree of success when applied to other problems in statistical physics which involve a system with no preferred length scale. However, in applying RG to turbulence we are faced with several technical difficulties. Indeed, as we shall argue in the following section, the deterministic nature of the Navier-Stokes equation (NSE) prevents us from rigorously performing one of the fundamental steps required in any RG calculation.

Bearing these points in mind, we have recently been using the results of numerical simulations in order to investigate the overall feasibility of applying RG to the problem of Navier-Stokes turbulence. Initial results from these simulations were previously presented at the 11th Turbulent Shear Flow colloquium in Grenoble (McComb et al., 1997). The current paper details the continuation of this work, both in order to probe the fundamental aspects involved in applying RG to turbulence, and to specifically probe the validity of uncontrolled approximations used in the theory. It further describes the results obtained when using the RG calculation to provide the subgrid model for a LES. We begin by outlining the RG approach and its difficulties.

RENORMALIZATION GROUP THEORY

RG provides us with a method for dealing with problems involving a large range of length or time scales, such as occur when considering critical phenomena or turbulence, and involves, in principle at least, a relatively simple procedure. First, we eliminate a range of the shortest wavelengths in the system, their effect upon the remaining scales being retained in average form as a contribution to a transport coefficient. Second, we rescale the new system so that it is defined on the same interval as our original system. These two steps are then repeated until the system becomes invariant under the transformations, at which stage it is said to have reached a **fixed point**. This corresponds to the system having become insensitive to changes of scale (i.e. it has no preferred length scale), as we would expect to be the case within the inertial range of turbulence.

RG Applied to Turbulence

In order to apply RG to turbulence we work in Fourier wavenumber ($\mathbf{k}$) space and consider incompressible fluid turbulence, as governed by the solenoidal NSE

$$(\partial_t + \nu_0 k^2)u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t) \qquad (1)$$

where ν_0 is the kinematic viscosity of the fluid,

$$M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1}[k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})] \quad (2)$$

and the projector $D_{\alpha\beta}(\mathbf{k})$ is expressed in terms of the Kronecker delta $\delta_{\alpha\beta}$ as

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\mathbf{k}|^{-2} \quad (3)$$

Further to this, we also restrict our attention to stationary, isotropic, homogeneous turbulence, with dissipation rate ε and zero mean velocity, in which case we may write the pair correlation of velocities as

$$\langle u_\alpha(\mathbf{k},t) u_\beta(\mathbf{k}',t') \rangle = Q(k, t-t') D_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k}+\mathbf{k}') \quad (4)$$

where $Q(k, t-t')$ is the spectral density.

In order to apply RG to this system we first define a maximum cut-off wavenumber k_0 via the dissipation integral

$$\varepsilon = \int_0^\infty dk\, 2\nu_0 k^2 E(k) \simeq \int_0^{k_0} dk\, 2\nu_0 k^2 E(k) \quad (5)$$

where $E(k) = 4\pi k^2 Q(k)$ and $Q(k) = Q(k,0)$, so that k_0 is of the same order of magnitude as the Kolmogorov dissipation wavenumber k_d. We then filter the velocity field at $k = k_1$ according to

$$u_\alpha(\mathbf{k},t) = \begin{cases} u_\alpha^-(\mathbf{k},t) \text{ for } 0 < k < k_1 \\ u_\alpha^+(\mathbf{k},t) \text{ for } k_1 < k < k_0 \end{cases} \quad (6)$$

where $k_1 = (1-\eta)k_0$ and the bandwidth parameter satisfies the condition $0 < \eta < 1$. This allows us to rewrite (1) as individual evolution equations for the u^- and u^+ modes,

$$L_0 u_k^- = M_k^-(u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+) \quad (7)$$

$$L_0 u_k^+ = M_k^+(u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+) \quad (8)$$

where $L_0 = \partial_t + \nu_0 k^2$, and, for simplicity, all vector indices and independent variables are contracted into a single subscript.

Following the RG algorithm, our first step is to eliminate the u^+ modes in (7) by solving for their mean effect upon the remaining u^- modes. This results in an increment to the viscosity, i.e. $\nu_0 \to \nu_1 = \nu_0 + \delta\nu_0$. We then rescale the basic variables so that the 'new' NSE, defined on $0 < k < k_1$, looks like the original NSE for $0 < k < k_0$. A typical approach to the first of these steps is to eliminate the high-k modes by directly substituting the solution of (8) for each u^+ term in (7). However, problems are then encountered because of coupling between the u^- and u^+ modes. Even if we succeed in doing this, we immediately have the further problem of averaging out the u^+ modes. As described by Wilson (1975), such an average requires us to average over the u^+ whilst holding the u^- modes constant. Since the NSE is deterministic, such an operation clearly cannot be performed in a rigorous manner, as fixing the u^- also serves to constrain the u^+. To circumvent this problem, the idea of a **conditional average** was introduced (McComb et al., 1992a), based on the ansatz that a small uncertainty in the u^- modes will generate a large uncertainty in the u^+ modes. It has recently been shown that such an assumption is not without foundation (Machiels, 1997).

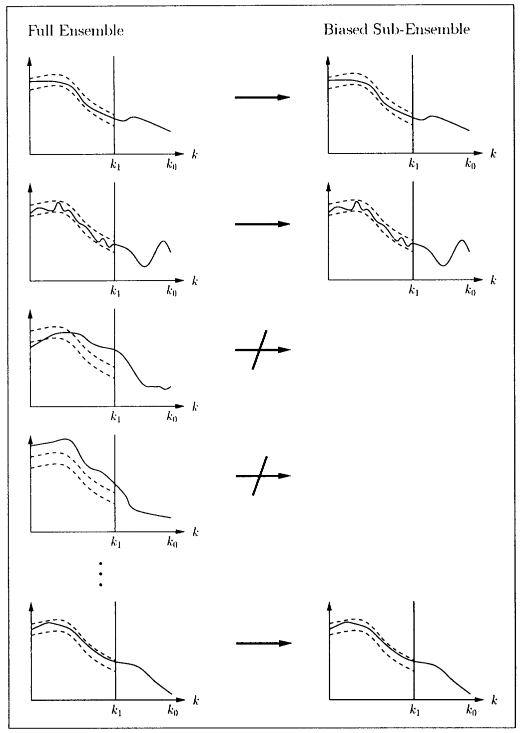

Figure 1: Schematic illustration of how the members of the biased sub-ensemble (RHS) are selected from the full ensemble (LHS). The ordinate represents the value of any variable we are using to define 'closeness' to a reference, such a reference being, for instance, the real part of a particular component of the velocity field.

RG Incorporating a Conditional Average

The basic idea of the conditional average is illustrated in Figure 1. Essentially, from the complete turbulent ensemble we select all those members whose u^- modes are 'close' to those of a particular reference member (those members whose low-k modes lie entirely within the two dashed lines in Fig. 1). These members form the so-called **biased sub-ensemble**. Whereas the ordinary ensemble average involves a summation over the members of the entire ensemble, the conditional average is defined as a summation over only those members of the biased sub-ensemble. In this way it is hoped that the conditional average of the u^- modes will give our reference field, whilst the conditional average of the u^+ modes will approach the result we would obtain if using an unconstrained ensemble.

Using such an conditional average, McComb and Watt (1992b) eliminated a band of high-k modes to obtain the result

$$\left(\partial_t + \nu_1(k)k^2\right) u_k^- = M_k^- u_j^- u_{k-j}^- \quad (9)$$

where $\nu_1(k) = \nu_0 + \delta\nu_0(k)$ and

$$\delta\nu_0(k) = -\frac{1}{k^2}\int d^3j\, Q_v^+(\mathbf{j}) \times \frac{2\mathbf{Tr}\left[M^-_{\alpha\beta\gamma}(\mathbf{k})M^+_{\gamma\rho\sigma}(\mathbf{k}-\mathbf{j})D_{\beta\sigma}(\mathbf{j})\right]}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2} \quad (10)$$

This band elimination was then extended to further shells as an RG calculation. By assuming that the effective viscosity and its increment scale in the same way (which is true at the fixed point) and that the rate of energy transfer is renormalized, they obtained both the factor $k^{-5/3}$ in the energy spectrum and a prediction for the Kolmogorov constant of 1.60 ± 0.01, in good agreement with experiment, for the range of bandwidths $0.25 < \eta < 0.45$.

Recently we have applied a new approach to the problem, utilising the idea of a conditional average which displays asymptotic freedom as we approach k_0 (McComb & Johnston, 1999). This calculation removes several criticisms applicable to the earlier theory, and regains the same result for the viscosity increment, but there still remain two uncontrolled approximations, the validity of which are unknown. First, we have to arbitrarily truncate the moment expansion at low order. Second, on the physical basis that u^- modes evolve slowly on timescales defined by the viscous timescale $(\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2)^{-1}$, we perform an integral using, as an approximation, a Taylor series truncated at zeroth order.

Along with other, more fundamental, questions regarding the applicability of RG, we have attempted to test the validity of these assumptions using numerical simulations. Some results from these studies are detailed in the following sections.

Application of RG to Stirred Hydrodynamics

The approach we have taken is not the only way in which RG may be applied to the NSE. An alternative approach was taken by Forster, Nelson and Stephen (1977), henceforth FNS, who considered the problem of the long wavelength properties of stirred hydrodynamics by restricting themselves to wavenumbers far below the inertial range and considering stirring forces which were multivariate normal. However the FNS approach is not immune to the problem of the determinism of the NSE, as was noted by Eyink (1994).

The technique of conditional averaging can be applied to answer this criticism. By defining at the level of forcing the constraint on the members of the sub-ensemble, we can find the necessary corrections to the velocity field, and their resulting effect on the renormalization of the viscosity. Doing this, we recover the results of FNS to second order in perturbation theory, whilst providing a firmer basis for the averaging operation.

NUMERICALLY TESTING THE THEORY

Our numerical simulations fall into two broad categories, DNS and LES, all of which were performed on a parallel machine, the Cray T3D, which has 512 processors and 64Mb of memory per processor. The DNS were performed at a resolution of 256^3 whilst the LES used a resolution of 128^3. As an example of the speed of these calculations, the DNS required approximately 0.5 CPU-hours per time step.

Both types of simulation followed the same well established method. For the DNS we follow the work of Orszag (1969) in constructing our initial velocity field, performing time integration using a second-order Runge-Kutta method, and achieving partial dealiasing by way of a random shifting method (see, for example, Rogallo, 1981). The LES code is simply a modification of that used in the DNS, the molecular viscosity being replaced by a wavenumber dependent eddy viscosity.

Results From DNS

Investigations which used DNS to probe the ideas behind the conditional average have been previously presented (McComb et al., 1997). Here a DNS with Taylor-Reynolds number $R_\lambda = 190$ was used to generate an ensemble of 1D-realizations. These were then used to study the statistical properties of both constrained and unconstrained sub-ensembles, considering correlations between realizations and the probability distribution functions of velocity increments. Although these results were preliminary in nature, they offered crucial support to the hypothesis that a conditional average may be used to reduce the number of degrees of freedom. More recently, we have been looking at using the results of DNS to probe particular assumptions made within the RG theory, and have further considered the applicability of the RG results by performing an LES using the predicted eddy viscosity.

Validity of the Approximations Made

As previously mentioned, aside from the more fundamental questions regarding the applicability of RG, we have also investigated the validity of the two uncontrolled approximations using the results from DNS. Our studies regarding the validity of assuming that the u^- modes evolve slowly when compared to the viscous timescale are still at a very preliminary stage and so are not reported here. However, our investigations regarding the arbitrary truncation of the moment expansion offer support to this aspect of the calculation.

The truncation may be justified via the introduction of a **local** Reynolds number $R(k_1)$, based on a length scale k_1^{-1}, the moment expansion being re-expressed as a power series in this parameter. The local Reynolds number can be expressed in terms of the energy spectrum as

$$R(k_1) = \frac{1}{\nu_0}\left(\frac{E(k_1)}{2\pi k_1}\right)^{1/2} \quad (11)$$

Hence, substituting the energy spectrum from a DNS we can get an estimate of its magnitude for any choice of cutoff k_1. The results from such a substitution are illustrated in Figure 2. As can be easily seen, this calculation yields the result that $R(k_1)$ is less than unity for any value of k_1 greater than $0.45k_d$, its value being approximately 0.1 as k_1 tends to k_d. Given these results, it would thus seem a reasonable approximation to neglect higher order terms in the moment expansion.

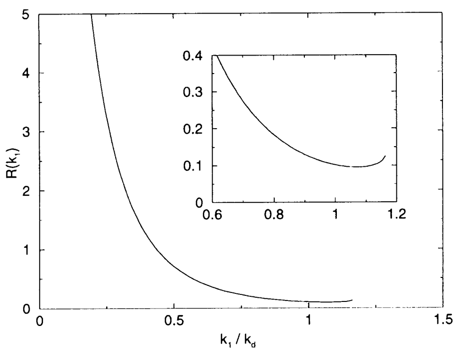

Figure 2: Prediction for the local Reynolds number $R(k_1)$ as found using the average energy spectrum from a 256^3 DNS with $\varepsilon = 0.149$, $\nu_0 = 10^{-3}$ and $R_\lambda = 190$. The inset illustrates the high wavenumber region of the graph.

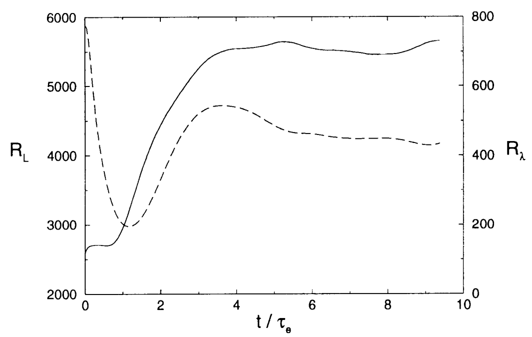

Figure 3: Time evolution of the integral length scale Reynolds number R_L (——) and Taylor-Reynolds number R_λ (-----) obtained from a 128^3 LES with $k_c = 60$, $\varepsilon = 0.274$ and $\nu_0 = 2 \times 10^{-4}$.

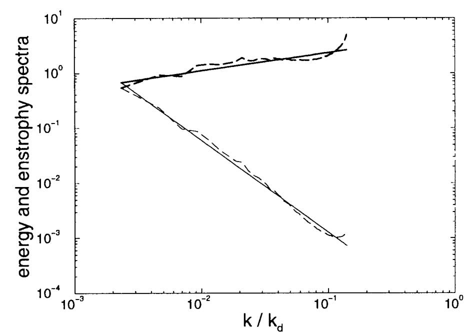

Figure 4: Time-averaged energy (-----) and enstrophy (-----) spectra obtained from a 128^3 LES with $k_c = 60$, $\varepsilon = 0.274$ and $\nu_0 = 2 \times 10^{-4}$. For comparison, the Kolmogorov energy (——) and enstrophy (——) are also plotted.

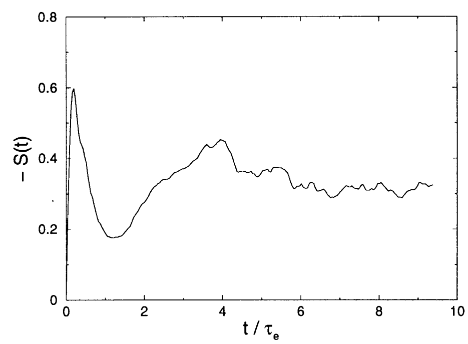

Figure 5: Time evolution of the skewness in a 128^3 LES with $k_c = 60$, $\varepsilon = 0.274$ and $\nu_0 = 2 \times 10^{-4}$.

Using RG to Provide a Model for LES

We have performed our LES at a resolution of 128^3 with a maximum resolved wavenumber, $k_c = 60$. We chose an energy input rate of $\varepsilon_w = 0.274$ and a kinematic viscosity $\nu_0 = 2 \times 10^{-4}$, giving a value for the Kolmogorov dissipation wavenumber of $k_d \simeq 430$. From Figure 3, we can see that once we reach a fully evolved state we achieve a Taylor-Reynolds number of the order of 400.

We begin with the results for energy and enstrophy spectra, plotted in Figure 4. These results have been time-averaged over the final five eddy-turnover times of the simulation. It will be noticed that there is a turn-up of the spectra at the high-k end, however inspections of the single-time energy spectra show no evidence that the energy in this region accumulates over time.

Figure 5 shows the development of the skewness in our simulations. It can be seen that this varies between -0.3 and -0.4 once the system is fully evolved. This is smaller in magnitude than the value of approximately -0.5 that we would expect to see from a DNS (see, for instance, Wang et al., 1996) but, in an LES, one loses the detail of the small scales, which give rise to a large proportion of the skewness (Dubois et al., 1997).

For illustration, we have also plotted a graph showing a measure of the isotropy in each wavenumber shell (Curry et al., 1984), Figure 6, along with a graph showing transport power and dissipation rate, Figure 7. The latter is of interest because it is a criterion for the existence of an inertial range that these should reach the same value.

If we consider the results from lower resolution (32^3) simulations shown in Figures 8 and 9, we can also see that the RG eddy viscosity gives results which compare well with both those obtained using an alternative model for the eddy viscosity, that obtained by Kraichnan (1976) using the direct interaction approximation, and also those which we obtain from a DNS with the same parameters.

For these low resolution simulations, we also have available plots of the vorticity fields, Figures 10, 11 & 12. As can be seen here, there is little qualitative difference between the plots for any of the simulations, both LES models giving results comparable to that which we obtain by taking the results of a 256^3 DNS and truncating to 32^3, hence removing the small scales.

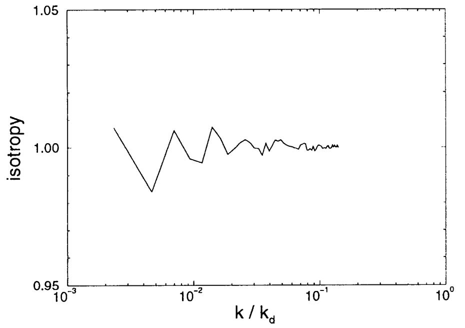

Figure 6: A measure of the isotropy obtained from a 128^3 LES with $k_c = 60$, $\varepsilon = 0.274$ and $\nu_0 = 2 \times 10^{-4}$. A value of close to unity indicates that the velocity field is approximately isotropic at that point.

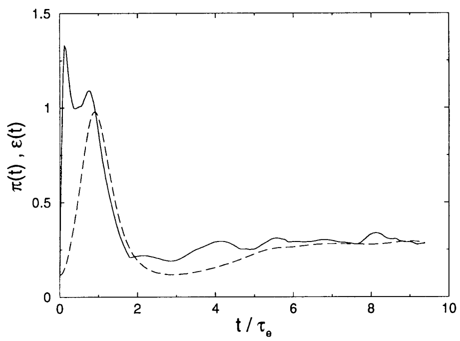

Figure 7: Time evolution of the transport power π (——) and dissipation rate ε (-----) as obtained from a 128^3 LES with $k_c = 60$, $\varepsilon = 0.274$ and $\nu_0 = 2 \times 10^{-4}$.

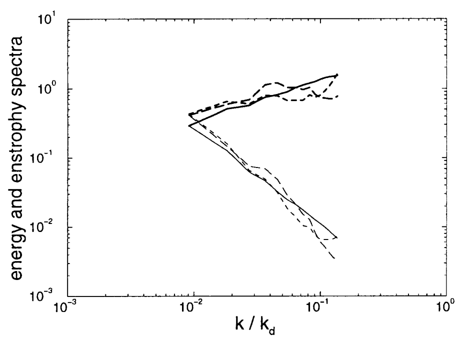

Figure 8: Time-averaged energy and enstrophy spectra obtained from a 256^3 DNS (energy ——, enstrophy ——) in comparison to a 32^3 LES using the RG eddy viscosity (energy -----, enstrophy -----) and a 32^3 LES using Kraichnan's TFM eddy viscosity (energy – – –, enstrophy – – –). All the simulations use the parameters $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$.

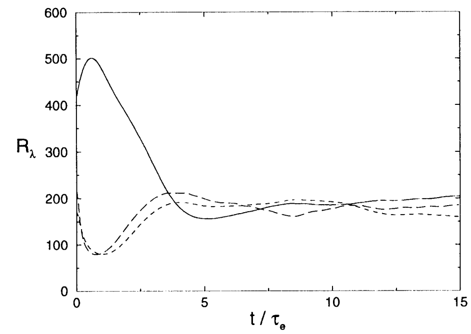

Figure 9: Time evolution of the Taylor-Reynolds number obtained from a 256^3 DNS (——), a 32^3 LES using the RG eddy viscosity (-----) and a 32^3 LES using Kraichnan's TFM (– – –). All the simulations use the parameters $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$.

CONCLUSIONS

Although not fully conclusive, since a 256^3 DNS has a very limited inertial range (see, for example, Yeung and Zhou, 1997), our results offer support both to the hypothesis of using a conditional average to eliminate turbulent modes, and to our truncation of the moment expansion. Further, an LES performed using the eddy-viscosity from our RG calculation gives results in good agreement with those obtained using alternative models, for instance Kraichnan's TFM (1976), and also the results of a DNS with the same parameters. The fact that this is so gives further support to our analytic calculations.

Our work is continuing in a similar direction as regards using DNS to study the remaining uncontrolled approximations arising in our analytic work. We intend to extend the reported LES work to study further the statistical properties of the resulting velocity field.

The main part of our work has concentrated on the chaotic part of the subgrid stress, which can be averaged out in terms of an effective viscosity, and which controls the dissipation rate of the LES. We are also now working on ways of handling the short-range (in wavenumber) coherent part of the subgrid stress, which is responsible for phase correlations between resolved and subgrid scales. On the basis of the deterministic connection involved here, we have experimented with an operational procedure to modify the behaviour of the resolved field in the neighbourhood of the cutoff wavenumber and some interesting preliminary results have been reported (Young & McComb, 1999).

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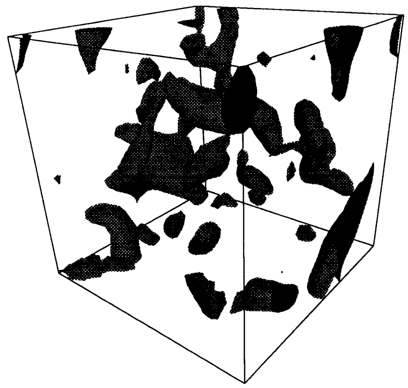

Figure 10: The vorticity field from a 32^3 LES using the RG eddy viscosity with $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$. The plotted iso-surfaces are for a value of 55% of the maximum vorticity.

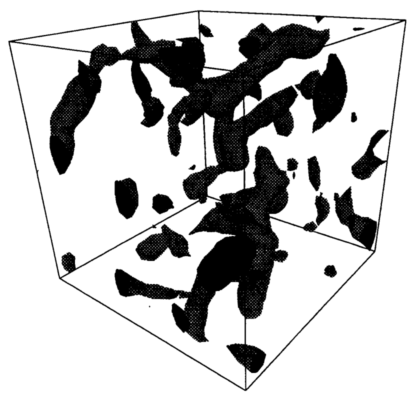

Figure 11: The vorticity field from a 32^3 LES using Kraichnan's TFM eddy viscosity with $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$. The plotted iso-surfaces are for a value of 55% of the maximum vorticity.

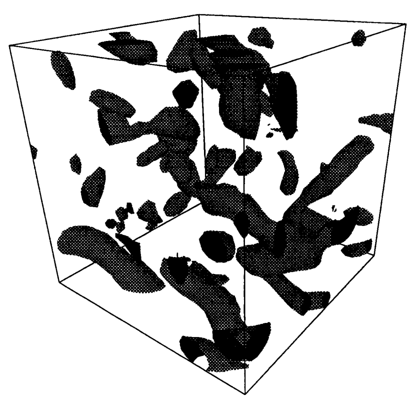

Figure 12: The vorticity field from a truncated 256^3 DNS with $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$. The plotted iso-surfaces are for a value of 55% of the maximum vorticity.

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