# ASSESSMENT OF INHOMOGENEITY EFFECTS ON THE PRESSURE TERM USING DNS DATABASE: IMPLICATIONS FOR RANS MODELS


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## ABSTRACT

The assumptions underlying the elliptic relaxation model for the pressure term in the Reynolds stress transport equations are examined through the analysis of a channel flow DNS database. The model is shown to be consistent with the data, in particular with regard to the evaluation of the length scale. Some features not accounted by the model, such as the asymmetry in the inhomogeneous direction of the two-point correlation function, are found to be responsible for the observed spurious amplification of the return to isotropy in the log layer. The expected reduction is obtained in the proposed new formulations of the elliptic relaxation equation. The common belief that this reduction is due to the wall echo effect is shown to be erroneous.


## INTRODUCTION

One of the most important and difficult tasks for turbulence modelers is to model the pressure term in the Reynolds stress transport equations, since it is the most significant unclosed term. Following the pioneering work of Chou (1945), this term is commonly split into rapid, slow and surface parts. The latter is usually neglected or represented by the so-called *wall echo* terms, while the rapid part is modeled by introducing a fourth order tensor, considering that the length scale of the variations of the velocity gradient is large in comparison with that of the two-point correlations.

This approach, which leads to the loss of the non-local nature of the pressure term, was found by Bradshaw *et al.* (1987) to be valid only for $y^+ \geq 40$ in a channel flow at $Re_\tau = 180$. Therefore, the influence of the wall on turbulence cannot be reproduced without introducing correction terms to this type of models. In order to avoid such *ad hoc* modifications, Durbin (1991) proposed to model the two-point correlations in the integral equation for the pressure term by an exponential function. This novel method leads to the so-called *elliptic relaxation equation*, which reproduces the non-local effect and enables the derivation of second moment closure models integrable down to solid boundaries.

While the elliptic relaxation model has led to very encouraging results, some issues remain open and room for improvement exists. The present work aims to assess the validity of some model assumptions through the analysis of a DNS database, which has never been done before. Particular attention will be focused on the shape of the correlation function as well as the length scale used in the model. New formulations, which take into account the variation of the length scale and the anisotropy of the correlation function, will be proposed, and their behavior in the log layer evaluated.

## PRESENTATION OF THE PROBLEM

### The elliptic relaxation approach

The pressure term entering the Reynolds stress transport equations is

$$\rho\phi_{ij} = -\left(\overline{u_j p_{,i}} + \overline{u_i p_{,j}}\right) \qquad (1)$$

The gradient of the pressure fluctuation, as well as the pressure fluctuation itself, satisfy the Poisson equation

$$\nabla^2 p_{,k} = -\rho\left(2U_{i,j}\, u_{j,i} + u_{i,j}\, u_{j,i} - \overline{u_{i,j}\, u_{j,i}}\right)_{,k} \qquad (2)$$

Eq. (2) is assumed to satisfy the boundary condition $p_{,kn} = 0$, where $\mathbf{n}$ denotes the unit vector normal to the wall. This assumption is equivalent to requiring that the "Stokes part" of the pressure gradient, namely the part produced by the inhomogeneous boundary condition, be negligible (Kim, 1989). The general solution of Eq. (2) is

$$p_{,k}(\mathbf{x}) = \int_\Omega \nabla^2 p_{,k}(\mathbf{x}') G_\Omega(\mathbf{x}, \mathbf{x}')\, dV(\mathbf{x}') \qquad (3)$$

where $G_\Omega$ is the Green function of the domain, i.e. the solution of Eq. (2) in which the RHS is replaced by the Dirac function $\delta(\mathbf{x}' - \mathbf{x})$. Eq. (3) does not contain any surface term because the Green function satisfies the same homogeneous Neumann boundary condition as for $p_{,k}$.

The integral equation of the pressure term is a consequence of Eq. (3):

$$\rho\phi_{ij}(\mathbf{x}) = \int_\Omega \Psi_{ij}(\mathbf{x},\mathbf{x}')G_\Omega(\mathbf{x},\mathbf{x}')dV(\mathbf{x}') \qquad (4)$$

where $\Psi_{ij}(\mathbf{x},\mathbf{x}') = -\overline{u_j(\mathbf{x})\nabla^2 p_{,i}(\mathbf{x}')} - \overline{u_i(\mathbf{x})\nabla^2 p_{,j}(\mathbf{x}')}$.

Durbin (1991) proposed to model this two-point correlation using the following definition of the correlation function $f(\mathbf{x},\mathbf{x}')$:

$$\Psi_{ij}(\mathbf{x},\mathbf{x}') = \Psi_{ij}(\mathbf{x}',\mathbf{x}')f(\mathbf{x},\mathbf{x}') \qquad (5)$$

and to approximate the correlation function by

$$f(\mathbf{x},\mathbf{x}') = \exp(-r/L) \qquad (6)$$

where $r = \|\mathbf{x}' - \mathbf{x}\|$. Note that the correlation function cannot depend on the component $(i,j)$ in order to preserve the tensorial properties of the pressure term. In a *free space*, the Green function is simply $G_{\mathbf{R}^3}(r) = -1/4\pi r$. Using Eqs. (5) and (6) in Eq. (4) then yields a convolution product between $-\Psi_{ij}$ and $E_{\mathbf{R}^3}(r) = \exp(-r/L)/4\pi r$, which is the Green function associated with the Yukawa operator $-\nabla^2 + 1/L^2$. Hence, the modeled pressure term satisfies the elliptic relaxation or Yukawa equation:

$$\phi_{ij} - L^2\nabla^2\phi_{ij} = \phi_{ij}^h \qquad (7)$$

In Eq. (7), the original RHS, $-\rho^{-1}L^2\Psi_{ij}(\mathbf{x},\mathbf{x})$, has been replaced by a quasi-homogeneous model $\phi_{ij}^h$, such as IP or SSG model, noting that in homogeneous situations $\phi_{ij}$ must relax to $\phi_{ij}^h$. This equation can easily be generalized in the case of a channel flow (Manceau *et al.*, 1998, 1999).

## Issues to examine

The elliptic relaxation approach is mainly based on the modeling by an exponential function of the correlation function $f(\mathbf{x},\mathbf{x}')$ defined in Eq. (5). This approximation was introduced intuitively by Durbin (1991) in order to preserve the non-local effect on the pressure term. In the present study, the two-point correlation $\Psi_{ij}(\mathbf{x},\mathbf{x}')$ will be evaluated from the DNS database of a channel flow at $Re_\tau = 590$ (Moser *et al.*, 1999), to assess the shape of the correlation function and the validity of Durbin's approximation. In addition, the length scale involved in Eqs. (6) and (7) will be evaluated, in order to validate the use of the turbulent length scale $k^{3/2}/\varepsilon$ in the main part of the flow and the Kolmogorov length scale in the near-wall region.

The ultimate objective of the present DNS analysis is to find ways to improve the behavior of the model. As pointed out by Wizman *et al.* (1996), the elliptic relaxation equation does not act in the right direction in the log layer. Indeed, the function $\phi_{ij}^h$, like $\varepsilon$, behaves in this region as $1/y$. If the length scale used in the model is $L = C\kappa y$, the solution of Eq. (7) is

$$\phi_{ij} = (1 - 2C^2\kappa^2)^{-1}\phi_{ij}^h \qquad (8)$$

which results in an amplification of the return to isotropy, instead of the expected reduction.

Wizman *et al.* (1996) proposed to modify the elliptic relaxation equation, in order to correct this behavior. For instance, they introduced the so-called *neutral formulation*, replacing in Eq. (7) the term $L^2\nabla^2\phi_{ij}$ by $\nabla^2(L^2\phi_{ij})$. This formulation exhibits neither amplification nor reduction in the log layer. However, it suffers from a lack of justification.

The present work attempts to provide a more solid basis to develop other new formulations. The central idea is that the correlation function cannot be approximated by an isotropic exponential function. Indeed, there is no basis to assume that the two-point correlations between the velocity and the Laplacian of the pressure gradient should have this feature, given that the contours of the two-point correlations between velocity components are packed between the point of zero separation and the wall (Sabot, 1976).

Furthermore, the erroneous model behavior in the log layer leads to difficulties in predicting accurately both the viscous sublayer and log layer, and hence compromises are needed to calibrate the coefficients. This limits the influence of the elliptic relaxation to a region very close to the wall. Some improvements can be expected by extending this influence to a larger region.

# RESULTS AND DISCUSSION

## The wall echo effect

In a semi-infinite space, bounded by a plane, the Green function is $G_\Omega(\mathbf{x},\mathbf{x}') = -1/4\pi r - 1/4\pi r^*$, where $r^* = \|\mathbf{x}'^* - \mathbf{x}\|$, $\mathbf{x}'^*$ being the image point of $\mathbf{x}'$ in the wall. It has been widely accepted that this image term, representing the wall echo effect, is responsible for the reduction of the pressure term. This concept has led to the inclusion of Gibson & Launder type wall echo terms.

However, since this term appears in the Green function with the same sign as the principal term, it actually *increases* the pressure fluctuation and hence the pressure term. Therefore, *the wall echo cannot be responsible for the damping of the energy redistribution.*

In a channel, the exact Green function is known only after taking Fourier transforms in homogeneous directions, which is not relevant in the context of this work. Nevertheless, it can be approximated (Manceau *et al.*, 1998, 1999) by $H(\mathbf{x},\mathbf{x}'_0) = -1/4\pi r_{-1} - 1/4\pi r_0 - 1/4\pi r_1$, with $r_n = \|\mathbf{x}'_n - \mathbf{x}\|$, where $\mathbf{x}'_{-1}$ and $\mathbf{x}'_1$ are the image points of $\mathbf{x}'_0$ with respect to each wall.

When $G_\Omega \simeq H$ is used, Eq. (4) becomes

$$\rho\phi_{ij}(\mathbf{x}) = -\int_\Omega \frac{\Psi_{ij}(\mathbf{x},\mathbf{x}'_0)}{4\pi}\left(\frac{1}{r_{-1}} + \frac{1}{r_0} + \frac{1}{r_1}\right)dV(\mathbf{x}'_0) \qquad (9)$$

Fig. 1 shows the three terms in the integrand of Eq. (9) in a channel at $y^+ = 30$. The image term arising from the far wall located at $y^+ = 1180$ is negligible, but the $n = -1$ term, which arises from the near wall at $y^+ = 0$ and has the same sign as the principal term, makes a significant, positive contribution to the integral (even though the $n = 0$ term goes to infinity at zero separation, the volume integral of $1/r$ between $r = 0$ and 1 is only $2\pi$).

Thus, the traditional way of modeling the damping of the return to isotropy through additional wall echo terms should be abandoned. It will be shown that this effect can be taken into account by a proper reformulation of the elliptic relaxation equation.

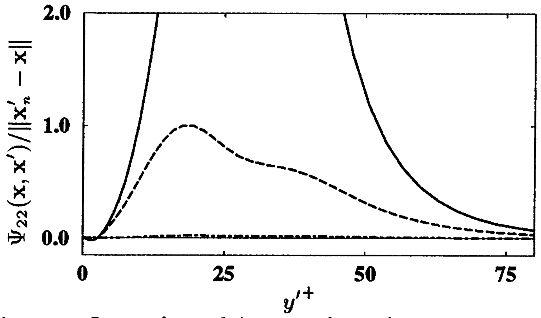

Figure 1: Comparison of the terms in the integrand of Eq. (9) ($i = j = 2$) at $y^+ = 30$: —— principal term ($n = 0$), --- image term due to the wall at $y'^+ = 0$ ($n = -1$), -·-·- image term due to the wall at $y'^+ = 1180$ ($n = 1$). The normalization is such that the maximum of $n = -1$ term is 1.

## Asymmetry in $y$-direction

The correlation function defined by Eq. (5) is modeled by an isotropic exponential function in the elliptic relaxation method. Using DNS data, a correlation function $f(\mathbf{x}, \mathbf{x}')$ can be calculated for each component of $\phi_{ij}$ from

$$f(\mathbf{x}, \mathbf{x}') = \Psi_{\alpha\beta}(\mathbf{x}, \mathbf{x}')/\Psi_{\alpha\beta}(\mathbf{x}', \mathbf{x}') \tag{10}$$

without summation over Greek indices. It is thus impossible to derive a model of $f$ which matches the DNS results for all the components. Hence, the following analysis should be interpreted in a qualitative rather than quantitative sense.

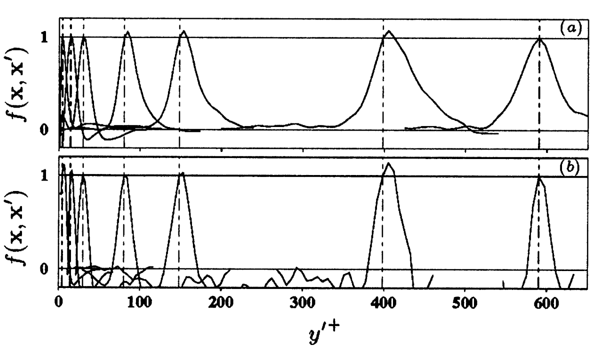

Figure 2: Correlation function calculated from the DNS data at different locations: $y^+ = 4$; $y^+ = 14$; $y^+ = 30$; $y^+ = 80$; $y^+ = 150$; $y^+ = 400$; $y^+ = 590$. $f$ is evaluated from Eq. (10) with ($a$) $\alpha = \beta = 1$; ($b$) $\alpha = \beta = 2$. Separations in $x$- and $z$-directions are zero.

Fig. 2 shows the correlation functions corresponding to $\phi_{11}$ and $\phi_{22}$ at different locations. Several observations can be made:

- The correlation length scale is larger at every location for the 11 component than for the 22 component. Hence, only a global accounting of the non-local effect is possible, which does not reproduce exactly the data.
- The correlation functions exhibit negative excursions. This calls into question the use of an exponential function to model them. However, this model will prove to be valid in the subsequent analysis.
- The correlation functions have clearly asymmetrical shapes, particularly in the log layer.

The last feature is the most important one. Indeed, when an isotropic correlation function is used, points between the fixed position and the wall are over-weighted. In the log layer, since the amplitude of $\Psi_{ij}$ decreases rapidly with distance to the wall, this results in the over-estimation of the amplitude of the pressure term described earlier. This appears to be the main improvable point of the model.

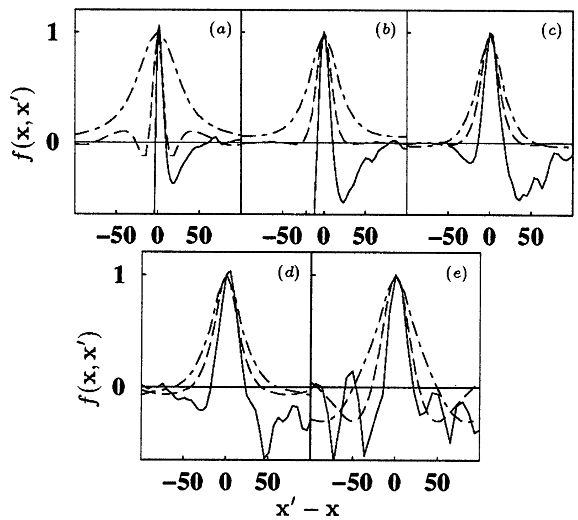

Figure 3: Shape of the correlation function defined by Eq. (10) with $\alpha = \beta = 2$, evaluated from the DNS, for separation in the 3 principal directions at different $y$ locations. ($a$) $y^+ = 14$, ($b$) $y^+ = 30$, ($c$) $y^+ = 80$, ($d$) $y^+ = 150$, ($e$) $y^+ = 590$. Separations: -·-·- $x$-direction, ($\Delta y = \Delta z = 0$); —— $y$-direction, ($\Delta x = \Delta z = 0$); --- $z$-direction, ($\Delta x = \Delta y = 0$).

## Anisotropy

Figs. 3a-e compare the correlation function corresponding to $\phi_{22}$ for separations in the 3 principal directions at five $y$ locations. It is observed that the distance of correlation is larger in the streamwise direction than in the other two directions. This feature is consistent with the streamwise elongation of turbulent structures observed in experiments. This anisotropy is most significant near the wall (Fig. 3a), and becomes less pronounced away from it (Figs. 3b-d), yet it still appears even in the center of the channel (Fig. 3e).

The above observation calls into question the use of the model given by (6), which does not distinguish among streamwise, spanwise and wall-normal directions. Although the anisotropy cannot be considered responsible for the model defects noted previously, since the non-local character has no effect in homogeneous directions, this feature may become important in more complex flows.

#### Length scales

The definition of the length scale used in the model (6) is not obvious. Indeed, the standard definition of the correlation function $f$ to be used in Eq. (4) is

$$\Psi_{ij}(\mathbf{x},\mathbf{x}') = \Psi_{ij}(\mathbf{x},\mathbf{x}) f(\mathbf{x},\mathbf{x}') \qquad (11)$$

where the one-point correlation is expressed in $\mathbf{x}$, the point where the velocities are evaluated. $\Psi_{ij}$ can then be taken outside the integral in Eq. (4). This formulation leads to the definition of

$$L^2_{\text{int}}(\mathbf{x}) = \left| \int_\Omega f(\mathbf{x},\mathbf{x}') G_\Omega(\mathbf{x},\mathbf{x}') dV(\mathbf{x}') \right| \qquad (12)$$

which is an integral scale, since the ratio between the integral and the correlation at zero separation can be written as

$$\rho\phi_{ij}(\mathbf{x}) = \pm L^2_{\text{int}}(\mathbf{x}) \Psi_{ij}(\mathbf{x},\mathbf{x}) \qquad (13)$$

However, this formulation leads to the loss of the non-local effect. In order to preserve it, the correlation function must be defined by Eq. (5). With this definition, the one-point correlation cannot be taken outside the integral. The length scale $L$ is then no longer an integral scale and thus cannot be evaluated by Eq. (13).

Nevertheless, $L$ is the integral from zero to infinity of $\exp(-r/L)$. Therefore, one may attempt to define a length scale in each direction by integrating $f(\mathbf{x},\mathbf{x}')$ along a line. But this method leads to a paradox in the homogeneous directions. For instance, the integral along the $x$-direction of the correlation function defined by Eq. (10) with $\alpha = \beta = 1$ gives exactly zero. This is due to the fact that $\int_0^\infty f(\mathbf{x},r)dr$ is very different from the correct definition of the integral scale, Eq. (12), which reduces to $\int_0^\infty r f(\mathbf{x},r)dr$ for isotropic turbulence in free space. Note also that, as a practical matter, $\int_0^\infty r f(\mathbf{x},r)dr$ cannot be used to evaluate $L$ either because the $r$ factor in the integrand tends to amplify the numerical noise at large separation where the true value of $f$ is small.

In the following, a very simple definition of the length scale will be used. It is noted that $\exp(-r/L)$ takes the value $1/e$ for $r = L$. Hence, one can define $L$ as the half-width of the correlation function between the two points at which $f = 1/e$. Notwithstanding its simplicity, this method enables the evaluation of the qualitative behavior of $L$ across the channel with directional dependence.

Fig. 4 shows a comparison of the different length scale definitions. These results are rather surprising and very encouraging. First, the correlation length scale $L$ is very close to the integral length scale $L_{int}$, except in the vicinity of the wall. This shows that small separations contribute most to the integral and that larger separations, which show a more complicated behavior including negative excursions in Fig. 2, can then be neglected. This result justifies the use of a simple exponential function to model the correlation function. Furthermore, Fig. 4 shows that in the main part of the flow, the correlation length scale $L$ can, in a standard way, be evaluated by the turbulent length scale $L_T = C_L k^{3/2} \varepsilon^{-1}$.

Secondly, in the near-wall region ($y^+ \leq 60$), the correlation length scale $L$ approaches the value of 6, whereas the integral scale $L_{int}$ decreases rapidly toward the wall. As shown in Fig. 4, in this region, $L$ behaves like the Kolmogorov length scale $L_K = C_\eta C_L \nu^{3/4} \varepsilon^{-1/4}$.

Finally, it can be seen that the point where $L$ and $L_{int}$ diverge is approximately where the turbulent length scale $L_T$ becomes smaller than the Kolmogorov length scale $L_K$. This justifies the use of the formulation $L = \max(L_K, L_T)$ in Durbin's model.

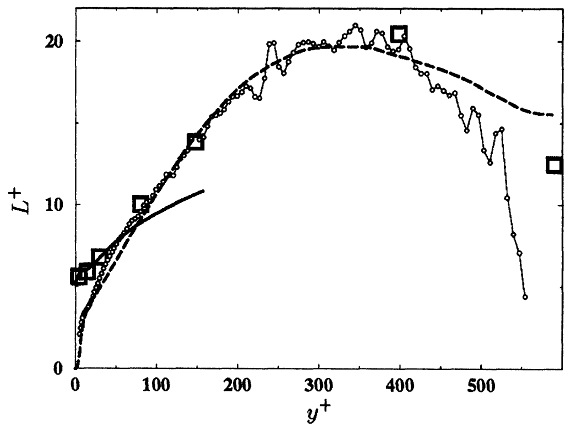


Figure 4: Comparison of the different length scales evaluated from the DNS: □ Length scale $L$ defined as the half-width of the correlation function $f$ shown in Fig. 2b; o——o Integral scale $L_{int}$ given by Eq. (13) with $i = j = 2$; – – – Turbulent length scale used in the model $L_T = C_L k^{3/2} \varepsilon^{-1}$ ($C_L = 0.045$); —— Kolmogorov length scale used near the wall in the model $L_K = C_\eta C_L \nu^{3/4} \varepsilon^{-1/4}$ ($C_\eta = 80$).

## NEW FORMULATIONS

#### Space transformation

As emphasized in the preceding sections, the elliptic relaxation equation does not behave correctly in the log layer. New formulations of this equation need to be derived, based on the results of the previous section.

The first problem to be noted is the rapid variation of the length scale $L$ across the channel. Indeed, the model is derived from Eq. (4) using Eqs. (5) and (6), which yields, in a free space,

$$\rho\phi_{ij}(\mathbf{x}) = -\int_\Omega \Psi_{ij}(\mathbf{x}',\mathbf{x}') \frac{\exp(-r/L)}{4\pi r} dV(\mathbf{x}') \qquad (14)$$

This equation can be inverted to give Eq. (7) only if it is a convolution product, i.e., if $L$ is a constant, or if $L$ can be considered locally as a constant over a distance corresponding to the correlation length scale, which is $L$ itself. In the log layer, $\mathrm{d}L/\mathrm{d}y$ is about $\kappa = 0.41$, depending on the coefficient used in the model. Therefore, $L$ is not locally a constant and Eq. (7) is not rigorously the inverse of Eq. (14).

In order to take into account the variation of $L$, a coordinate transformation $\mathbf{x} \mapsto \boldsymbol{\alpha}(\mathbf{x})$ can be introduced, such that in the transformed space, the length scale is a constant, and the boundaries of the domain are preserved: $\alpha(\Omega) = \Omega$. In a channel, this transformation is simply given by $d\alpha_2/L_\alpha = dy/L(y)$, where $L_\alpha$ is a constant. Fig. 5 shows how it transforms the shape of the correlation function. The correlation length scale, which corresponds to the half-width of the correlation function, is then a constant.

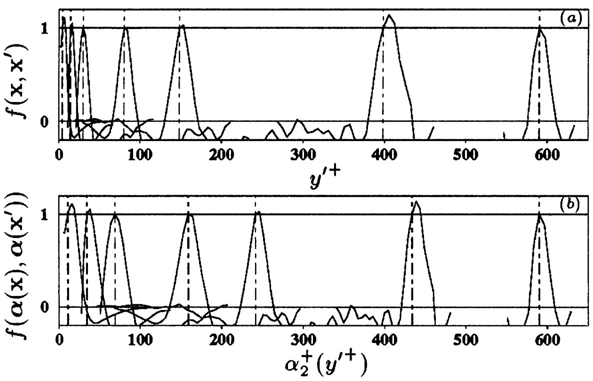


Figure 5: Effect of the space transformation on the correlation function: ($a$) Original correlation function (same as Fig. 2b); ($b$) Correlation function after transformation of $y^+$-axis. See Fig. 2 for additional caption.

If one defines functions $\xi_i = p_{,i} \circ \alpha^{-1}$, $w_i = u_i \circ \alpha^{-1}$, $\zeta_{ij} = \phi_{ij} \circ \alpha^{-1}$, and $g_i = \partial^2 \xi_i / \partial x_k \partial x_k$, the same inversion as that leading to Eq. (3) gives

$$\xi_k(\mathbf{x}) = \int_\Omega g_k(\mathbf{x}') G_\Omega(\mathbf{x}, \mathbf{x}')\, dV(\mathbf{x}') \qquad (15)$$

where $G_\Omega(\mathbf{x}, \mathbf{x}')$ is the same Green function as in Eq. (3), since the domain is preserved by the transformation $\alpha$. The same method leading to Eq. (14) can be used here to show that, in a free space:

$$\rho \zeta_{ij}(\mathbf{x}) = -\int_\Omega \Theta_{ij}(\mathbf{x}', \mathbf{x}') \frac{\exp(-r/L_\alpha)}{4\pi r}\, dV(\mathbf{x}') \qquad (16)$$

where $\Theta_{ij}(\mathbf{x}, \mathbf{x}') = -\overline{w_j(\mathbf{x}) g_i(\mathbf{x}')} - \overline{w_i(\mathbf{x}) g_j(\mathbf{x}')}$. The coordinate tranformation $\alpha$ is chosen such that $L_\alpha$ is a constant, so that Eq. (16) is now a convolution product. Therefore, $\zeta_{ij}$ satisfies the Yukawa equation:

$$\zeta_{ij}(\mathbf{x}) - L_\alpha^2 \frac{\partial^2 \zeta_{ij}(\mathbf{x})}{\partial x_k \partial x_k} = -\rho^{-1} L_\alpha^2 \Theta_{ij}(\mathbf{x}, \mathbf{x}) \qquad (17)$$

The equation satisfied by $\phi_{ij}$, derived from Eq. (17), involves the Jacobian matrix of the inverse transformation, $\mathbf{A} = \nabla \alpha^{-1}$. This enables the introduction of a matrix of length scales, by defining $A_{ij} = L_{ij}/L_\alpha$. For instance, one can define $L_{ij} = L\, \overline{u_i u_j}/k$, which gives

$$\phi_{ij} - L_{kl} L_{ml} \frac{\partial^2 \phi_{ij}}{\partial x_k \partial x_m} - L_{ml} \frac{\partial L_{kl}}{\partial x_m} \frac{\partial \phi_{ij}}{\partial x_k} = \phi_{ij}^h \qquad (18)$$

or simply $L_{ij} = \delta_{ij} L$, which gives

$$\phi_{ij} - L \nabla \cdot (L \nabla \phi_{ij}) = \phi_{ij}^h \qquad (19)$$

The first formulation distinguishes among the different directions, as was found necessary in the preceding analysis, but introduces many new terms. The second one is much simpler but the directional information is lost. Both formulations take into account the gradient of the length scale, which can be considered as a correction to the original model. However, the log layer analysis now leads to:

$$\phi_{ij} = (1 - \beta C^2 \kappa^2)^{-1} \phi_{ij}^h \qquad (20)$$

with $\beta = \overline{u_1 u_2}^2/k^2 + \overline{u_2 u_2}^2/k^2 \simeq 0.15$ for the first formulation and $\beta = 1$ for the second one. It can be noted that these formulations still result in an amplification of the return to isotropy, although it is less pronounced.

This problem can be traced to the shape of the correlation function shown in Fig. 5b. It is seen that the space transformation does not remove all the asymmetry in the $y$-direction (note that, in this figure, the transformation of the correlations is used, instead of the correlations of the transformation). In the next section, a new correlation function which takes into account this asymmetry will be proposed.

## New correlation function

In order to account for the asymmetry of the correlation function, the gradient of the length scale can be introduced by letting $f(\mathbf{x}, \mathbf{x}') = \exp(-r/(L + (\mathbf{x}' - \mathbf{x}) \cdot \nabla L))$. With this correlation function, shown in Fig. 6b, the prediction of the two-point correlation is improved, particularly between the fixed point $\mathbf{x}$ and the wall (cf. Fig. 6a), where the original correlation function causes an over-estimation.

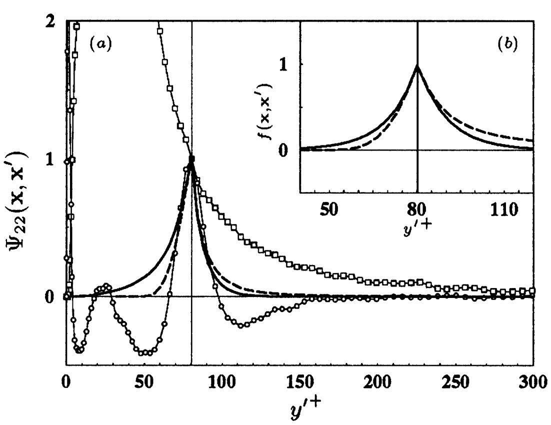


Figure 6: *A priori* test of the two-point correlation obtained at $y^+ = 80$ using two different correlation functions. ($a$) □ $\Psi_{22}(\mathbf{x}', \mathbf{x}')_{DNS}$; ○ $\Psi_{22}(\mathbf{x}, \mathbf{x}')_{DNS}$; —— $\Psi_{22}(\mathbf{x}, \mathbf{x}') = \Psi_{22}(\mathbf{x}', \mathbf{x}')_{DNS} \exp(-r/L)$; – – – $\Psi_{22}(\mathbf{x}, \mathbf{x}') = \Psi_{22}(\mathbf{x}', \mathbf{x}')_{DNS} \exp(-r/(L + (\mathbf{x}' - \mathbf{x}) \cdot \nabla L))$. ($b$) —— $\exp(-r/L)$; – – – $\exp(-r/(L + (\mathbf{x}' - \mathbf{x}) \cdot \nabla L))$.

Using a Taylor series expansion of the correlation function, it can be shown that the elliptic relaxation equation becomes

$$\left(1 + 16\, (\nabla L)^2\right) \phi_{ij} - L^2 \nabla^2 \phi_{ij} - 8 L \nabla L \cdot \nabla \phi_{ij} = \phi_{ij}^h \qquad (21)$$

The log layer analysis then yields

$$\phi_{ij} = (1 + 22\, C^2\kappa^2)^{-1}\phi_{ij}^h \qquad (22)$$

This new formulation exhibits now a reduction of the redistribution.

Additional correction terms can be derived, if second derivatives of the length scale are introduced into the correlation function. It can be shown that, at the first order, two extra terms appear, and the equation becomes

$$\left(1 - 4L\nabla^2 L + 20\,(\nabla L)^2\right)\phi_{ij} - L^2\nabla^2\phi_{ij} - 8L\nabla L\cdot\nabla\phi_{ij} = \phi_{ij}^h \qquad (23)$$

which gives, in the log layer, $\phi_{ij} = (1 + 26\, C^2\kappa^2)^{-1}\phi_{ij}^h$.

Thus, a simple modification of the model for the correlation function overcomes the deficiencies of the original model in the log layer, without introducing any "wall echo terms".

## CONCLUSIONS

A DNS database for a channel flow at $Re_\tau = 590$ has been used to assess the validity of the elliptic relaxation model and to understand how to improve its performance. Several conclusions can be drawn:

- The main approximation, which consists of modeling the correlation function by an exponential function, is consistent with the data. In particular, the similarity of the integral length scale $L_{int}$ and the correlation length scale $L$ shows that only the shape of the correlation function around zero separation is important to be modeled.
- The length scale introduced by Durbin, which is the turbulent length scale $C_L k^{3/2}\varepsilon^{-1}$ bounded near the wall by the Kolmogorov length scale $C_\eta C_L \nu^{3/4}\varepsilon^{-1/4}$, reproduces surprisingly well the evolution of the correlation length scale across the channel.
- The shape of the correlation function depends on the component of the two-point correlation tensor used to evaluate it. Only a global accounting of the non-local effect can then be expected.
- The image terms in the integral equation of the pressure term actually lead to an amplification of the return to isotropy. Hence, these terms cannot be considered responsible for the reduction of the redistribution.
- The spurious amplification of the redistribution in the log layer by the elliptic relaxation equation is due to the fact that the model does not take into account the strongly asymmetric shape of the correlation function. Simple modifications to the correlation function model lead to new formulations of the elliptic relaxation equation which do not present the same problem.
- The correlation function is anisotropic, elongated in the streamwise direction. Even though this feature has no effect in the case of a channel flow, it can affect more complex flows. The present analysis shows that it is possible to introduce directional information into the model, using a more complicated but tensorially correct formulation of the elliptic relaxation equation.

The physical insight gained through this study will enable further investigations to improve the elliptic relaxation method. The new formulations can be expected to give improved solutions, particularly in complex flow configurations. Their performance will be assessed through a number of test cases in future investigations.

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