ANISOTROPIC PRESTRESS THEORY FOR HOMOGENEOUSLY SHEARED TURBULENT FLOWS

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ABSTRACT

An isotropic prestress theory for the Reynolds stress, developed recently by Parks *et al.* (1998), is extended to include an anisotropic component of the turbulent prestress correlation induced by pressure fluctuations and fluctuations in the instantaneous Reynolds stress. A turbulent Deborah number, $De_t = \lambda\dot{\gamma}$, controls the long time response of the Reynolds stress subjected to a mean velocity gradient. A preclosure relaxation group, $N_R = \tau_V\dot{\gamma}$, controls the short time response of the Reynolds stress. For simple shear flows, the phenomenological theory predicts the existence of nonzero primary and secondary normal stress differences.

INTRODUCTION

Velocity fluctuations of a constant density, Newtonian fluid relative to an inertial frame of reference are governed by the following vector-valued equation (Monin and Yaglom, 1965)

$$-\langle L_V\rangle(\mathbf{u}') = \mathbf{F}' \tag{1}$$

where $\mathbf{F}' \equiv \mathbf{u}' \cdot \nabla\langle\mathbf{u}\rangle + \mathbf{f}'$ and

$$\mathbf{f}' \equiv \nabla \cdot \left[\frac{P'}{\rho}\underline{\underline{\mathbf{I}}} + \mathbf{u}'\mathbf{u}' - \langle\mathbf{u}'\mathbf{u}'\rangle\right] \tag{2}$$

The linear operator $\langle L_V\rangle(\cdot)$ defined by

$$\langle L_v\rangle(\mathbf{u}') \equiv \frac{\partial}{\partial t} + \langle\mathbf{u}\rangle \cdot \nabla - \nu\nabla^2 \tag{3}$$

accounts for transport of momentum fluctuations by mean convection and viscous stresses.

A statistically stationary representation for $\mathbf{u}'(\mathbf{x}, t)$ can be written in terms of a Green's function associated with the linear operator $\langle L_V\rangle(\cdot)$:

$$\mathbf{u}'(\mathbf{x},t) = -\int_{-\infty}^{t} d\hat{t} \int_{V} d\hat{V} \langle G_V\rangle(\mathbf{x},t \mid \hat{\mathbf{x}},\hat{t})\mathbf{F}'(\hat{\mathbf{x}},\hat{t}) \tag{4}$$

In a frame of reference moving with the local mean velocity, the Green's function is spatially peaked for $(t-\hat{t})\nu/\|\mathbf{x}-\hat{\mathbf{x}}\|^2 \ll 1$. For large differences in time, the Green's function relaxes to zero over the entire spatial domain (Morse and Feshbach, 1953); and, for an unbounded domain, $\langle G_v\rangle(\mathbf{x},t|\hat{\mathbf{x}},\hat{t})$ satisfies the following conservation property

$$\int_V d\hat{V} \langle G_v\rangle(\mathbf{x},t|\hat{\mathbf{x}},\hat{t}) = 1 \tag{5}$$

A formal representation of the kinematic Reynolds stress $\langle\mathbf{u}'\mathbf{u}'\rangle$ follows by multiplying Eq. (4) by $\mathbf{u}'$ and averaging the result:

$$\langle\mathbf{u}'\mathbf{u}'\rangle(\mathbf{x},t) = -\int_{-\infty}^{t} d\hat{t} \int_{V} d\hat{V} \langle G_V\rangle(\mathbf{x},t \mid \hat{\mathbf{x}},\hat{t}) \langle\mathbf{u}'(\mathbf{x},t)\mathbf{F}'(\hat{\mathbf{x}},\hat{t})\rangle \tag{6}$$

The components of the space-time correlation $\langle\mathbf{u}'(\mathbf{x},t)\mathbf{F}'(\hat{\mathbf{x}},\hat{t})\rangle$ relax to zero for large values of either $(t-\hat{t})/\tau_H$ or $\|\mathbf{x}-\hat{\mathbf{x}}\|/\ell_H$, where τ_H and ℓ_H represent characteristic temporal and spatial scales associated with the statistical space-time structure of the turbulence.

A *spatial smoothing approximation* that exploits the finite memory of the underlying turbulent structure and the relatively slow relaxation of the Green's function from its initial spatial delta distribution can be used to simplify the non-local representation of $\langle\mathbf{u}'\mathbf{u}'\rangle$ given by Eq. (6). Using this strategy, Parks *et al.* (1998) developed the following algebraic preclosure for the Reynolds stress:

$$<\mathbf{u}'\mathbf{u}'> = \underline{\underline{\mathbf{A}}}^T \cdot [\tau_v^2 <\mathbf{f}'\mathbf{f}'>] \cdot \underline{\underline{\mathbf{A}}} \tag{7}$$

where the dyadic-valued operator $\underline{\underline{\mathbf{A}}}$ is defined by

$$\underline{\underline{\mathbf{A}}}^{-1} = \underline{\underline{\mathbf{I}}} + \tau_v \nabla < \mathbf{u} > \tag{8}$$

The phenomenological time scale τ_V in Eq. (8) depends on the underlying temporal coupling between the space-time structure of the turbulence and the Green's function:

$$\tau_v = \int_{-\infty}^{t} d\hat{t}\, \mathbf{M}(\mathbf{x}, |t-\hat{t}|) \int_V d\hat{V} < G_v > (\mathbf{x}, t|\hat{\mathbf{x}}, \hat{t}) \tag{9}$$

Because the spatial integral of the Green's function is constant for an unbounded domain, the statistically stationary autocorrelation function $M(\mathbf{x}, |t-\hat{t}|)$ accounts for the temporal structure of the turbulence on τ_V. The foregoing *spatial smoothing ansatz* yields a preclosure theory that explicitly suppresses the spatial structure of the turbulence.

Eq. (7) relates the Reynolds stress to three distinct statistical properties of the turbulence: (1) the spatial gradient of the mean field; (2) the relaxation time τ_V; and, (3) the unclosed statistical correlation $\tau_v^2 <\mathbf{f}'\mathbf{f}'>$. As implied by Eq. (2), the prestress correlation $\tau_v^2 <\mathbf{f}'\mathbf{f}'>$ accounts for fluctuations in the instantaneous Reynolds stress and the pressure field. For homogeneous turbulent flows, the operator $\underline{\underline{\mathbf{A}}}$ reduces to the unit dyadic and the prestress correlation equals the Reynolds stress. Like the Reynolds stress, the prestress is a non-negative operator and, thereby, has three non-negative eigenvalues. Thus, it follows directly from Eq. (7) that *realizability* of the Reynolds stress (see, esp., Schumann, 1977) follows directly from the *realizability* of the turbulent prestress $\tau_v^2 <\mathbf{f}'\mathbf{f}'>$.

The prestress correlation can be written as the sum of an isotropic and an anisotropic operator:

$$\tau_v^2 <\mathbf{f}'\mathbf{f}'> = \frac{2}{3}\alpha\, \underline{\underline{\mathbf{I}}} + \underline{\underline{\tilde{\mathbf{H}}}} \tag{10}$$

The anisotropic component is symmetric and traceless, $\underline{\underline{\tilde{\mathbf{H}}}} = \underline{\underline{\tilde{\mathbf{H}}}}^T$ and $\mathrm{tr}(\underline{\underline{\tilde{\mathbf{H}}}}) = 0$. The analog of Eq. (10) for the Reynolds stress is

$$<\mathbf{u}'\mathbf{u}'> = \frac{2}{3} k\, \underline{\underline{\mathbf{I}}} + \underline{\underline{\tilde{\mathbf{b}}}} \tag{11}$$

where $\underline{\underline{\tilde{\mathbf{b}}}} = \underline{\underline{\tilde{\mathbf{b}}}}^T$ and $\mathrm{tr}(\underline{\underline{\tilde{\mathbf{b}}}}) = 0$. In Eq. (11), the turbulent kinetic energy k is defined by

$$k = \frac{1}{2}\mathrm{tr}(<\mathbf{u}'\mathbf{u}'>) \tag{12}$$

In Eq. (10), the isotropic prestress coefficient α is given by

$$\alpha = \frac{1}{2}\mathrm{tr}(\tau_v^2 <\mathbf{f}'\mathbf{f}'>) \tag{13}$$

Parks et al. (1998) completed the closure of Eq. (7) for homogeneously sheared turbulent flows (also see Parks, 1997) by assuming an isotropic prestress. Thus, with $\underline{\underline{\tilde{\mathbf{H}}}} = \underline{\underline{\mathbf{O}}}$, Eq. (7) becomes

$$<\mathbf{u}'\mathbf{u}'>^{IPS} = \frac{2}{3}\alpha\, \underline{\underline{\mathbf{A}}}^{-1} \cdot \underline{\underline{\mathbf{A}}} \tag{14}$$

Here the IPS-theory is extended phenomenologically to include an anisotropic component of the prestress that accounts for relaxation phenomenon (*i.e.*, return-to-isotropy) as well as first- and second-normal Reynolds stress differences for simple shear.

CLOSURE MODEL

Previous closure models for turbulence have focused directly on the anisotropic component of the Reynolds stress $\underline{\underline{\tilde{\mathbf{b}}}}$ (Monin and Yaglom, 1965; Speziale, 1991). The widely used, frame-indifferent Boussinesq approximation exemplifies these earlier approaches:

$$\underline{\underline{\tilde{\mathbf{b}}}} = -2\nu_e < \underline{\underline{\mathbf{S}}} > \tag{15}$$

In Eq. (15), ν_e is a scalar-valued eddy viscosity coefficient that depends on the local statistical state of turbulence and $< \underline{\underline{\mathbf{S}}} >$ is the mean strain rate. For large turbulent Reynolds numbers (*i.e.*, $k^2 >> \nu\varepsilon$), the k-ε theory assumes that $\nu_e = C_\mu k^2/\varepsilon$ where k and ε represent, respectively, the kinetic energy (see Eq. (12)) and the dissipation associated with velocity fluctuations:

$$\varepsilon \equiv \nu < (\nabla \mathbf{u}') : (\nabla \mathbf{u}')^T > \tag{16}$$

The dimensionless coefficient C_μ is constant for large turbulent Reynolds numbers. Transport equations for k and ε are used to compute local turbulent time and length scales within the flow domain (Launder *et al.*,1975; Hanjalic, 1994):

$$\begin{aligned} <\frac{D}{Dt}>(k) = &- <\mathbf{u}'\mathbf{u}'> : \nabla <\mathbf{u}> - \varepsilon \\ &+ \nabla \cdot [(\nu\underline{\underline{\mathbf{I}}} + C_k\, \tau_v <\mathbf{u}'\mathbf{u}'>) \cdot \nabla k] \end{aligned} \tag{17}$$

$$\begin{aligned} <\frac{D}{Dt}>(\varepsilon) = &- C_P \frac{<\mathbf{u}'\mathbf{u}'> : \nabla <\mathbf{u}>}{\tau_P} - C_D \frac{\varepsilon}{\tau_D} \\ &+ \nabla \cdot [(\nu\underline{\underline{\mathbf{I}}} + C_\varepsilon\, \tau_v <\mathbf{u}'\mathbf{u}'>) \cdot \nabla \varepsilon] \end{aligned} \tag{18}$$

The operator <D/Dt> is the substantial derivative relative to the mean velocity. For large turbulent Reynolds numbers, $\tau_p = \tau_D = k/\varepsilon$. The relaxation time τ_v , defined by Eq. (9), also scales with the turbulent turnover time, i.e., $\tau_v = C_R k/\varepsilon$. The dimensionless model parameters C_k , C_ε , C_P ,

C_D , and C_R are independent of the local statistical state of the turbulence.

For $\nu_e > 0$, Eq. (15) implies that the kinetic energy is irreversibly transferred from the mean field to the fluctuating field inasmuch as

$$-\langle\mathbf{u}'\mathbf{u}'\rangle : \nabla\langle\mathbf{u}\rangle = 2\nu_e \langle\underline{\underline{\mathbf{S}}}\rangle : \langle\underline{\underline{\mathbf{S}}}\rangle > 0 \qquad (19)$$

for all turbulent flows. Eq. (19) may partly justify the use of Eq. (15) as a model for the anisotropic stress. Unfortunately, however, Eq. (15) also predicts an equipartition of kinetic energy among the components of the fluctuating velocity in simple mean shear as well as a zero third invariant of the anisotropic stress (*i.e.*, III ≡ tr(**b·b·b**) =0). These unphysical results clearly misrepresent the underlying mechanism associated with the flux of momentum due to velocity fluctuations.

Parks et al. (1998) and Weispfennig et al. (1999) used Eq. (7) and an isotropic prestress (i.e., $\tilde{\underline{\underline{\mathbf{H}}}} = \underline{\underline{\mathbf{O}}}$) as an approximate model for the Reynolds stress. Although the isotropic prestress (ISP) theory predicts a positive first normal stress difference and a shear thinning eddy viscosity coefficient for simple mean shear, it erroneously predicts that the second normal stress difference is zero for simple shear flows.

Here, a nontrivial frame-indifferent relaxation model for the anisotropic prestress will be used to complete the closure of Eq. (7):

$$\tilde{\underline{\underline{\mathbf{H}}}} + \lambda[\dot{\underline{\underline{\mathbf{M}}}} - \frac{1}{3}\mathrm{tr}(\dot{\underline{\underline{\mathbf{M}}}})\underline{\underline{\mathbf{I}}}] = \beta\langle\underline{\underline{\mathbf{S}}}\rangle \qquad (20)$$

For large turbulent Reynolds numbers, the phenomenological parameters λ and β are assumed to scale with k and ε. Thus, $\lambda = C_\lambda k/\varepsilon$ and $\beta = 2kC_\beta k/\varepsilon$. C_λ and C_β are dimensionless constants independent of the local statistical state of the turbulence. In the above equation, $\dot{\underline{\underline{\mathbf{M}}}}$ is a frame indifferent time derivation defined by

$$\dot{\underline{\underline{\mathbf{M}}}}(\tilde{\underline{\underline{\mathbf{H}}}}) \equiv \langle\frac{D}{Dt}\rangle(\tilde{\underline{\underline{\mathbf{H}}}}) - \langle\underline{\underline{\mathbf{W}}}\rangle^T\cdot\tilde{\underline{\underline{\mathbf{H}}}} - \tilde{\underline{\underline{\mathbf{H}}}}\cdot\langle\underline{\underline{\mathbf{W}}}\rangle$$
$$+ C_M[\langle\underline{\underline{\mathbf{S}}}\rangle\cdot\tilde{\underline{\underline{\mathbf{H}}}} + \tilde{\underline{\underline{\mathbf{H}}}}\cdot\langle\underline{\underline{\mathbf{S}}}\rangle] \qquad (21)$$

$\langle\underline{\underline{\mathbf{S}}}\rangle$ and $\langle\underline{\underline{\mathbf{W}}}\rangle$ represent, respectively, the mean strain rate dyadic and the mean vorticity dyadic. With $C_M = 0$, Eq. (21) reduces to the corotational Jaumann derivative; for $C_M = +1$ and $C_M = -1$, Eq. (21) yields the upper and lower convected derivatives of Oldroyd, respectively (Bird *et al.* 1977; Denn, 1990; Joseph, 1990; Speziale, 1991). The operator $\dot{\underline{\underline{\mathbf{M}}}}(\cdot)$ is an objective property of the mean field for $-\infty < C_M < +\infty$. Physically, the Jaumann derivative of $\tilde{\underline{\underline{\mathbf{H}}}}$ represents the temporal changes in the anisotropic prestress relative to a frame of reference moving with the local mean velocity and rotating with an angular velocity equal to the mean vorticity. Note that Eqs. (20) and (21) preserve the symmetry and contraction properties of the anisotropic prestress, $\tilde{\underline{\underline{\mathbf{H}}}} = \tilde{\underline{\underline{\mathbf{H}}}}^T$ and $\mathrm{tr}(\tilde{\underline{\underline{\mathbf{H}}}}) = 0$. As shown momentarily, the foregoing APS-theory can account for the long-time relaxation response of the Reynolds stress to a sudden removal of the mean field gradient (*i.e.*, return-to-isotropy). The theory also predicts the existence of nonzero second normal stress differences in simple mean shear flows.

ANISOTROPIC DECAY

For homogeneous turbulence, $\tilde{\underline{\underline{\mathbf{H}}}} = \tilde{\underline{\underline{\mathbf{b}}}}$ and $\alpha = k$. Therefore, with $\tilde{\underline{\underline{\mathbf{H}}}} = 2k\underline{\underline{\mathbf{b}}}$, Eq. (20) can be written as

$$\frac{k}{\varepsilon}\frac{d}{dt}(\underline{\underline{\mathbf{b}}}) = -\frac{(1-C_\lambda)}{C_\lambda}\underline{\underline{\mathbf{b}}} \qquad (22)$$

and Eq. (17) becomes

$$\frac{1}{k}\frac{dk}{dt} = -\frac{\varepsilon}{k} \qquad (23)$$

With the second invariant of the anisotropic stress defined as $\mathrm{II} = \mathrm{tr}(\underline{\underline{\mathbf{b}}}\cdot\underline{\underline{\mathbf{b}}})$, it follows directly from Eq. (22) that

$$\frac{k}{\varepsilon}\frac{d(\mathrm{II})}{dt} = -\frac{2(1-C_\lambda)}{C_\lambda}\mathrm{II} \qquad (24)$$

Eqs. (23) and (24) imply that

$$\mathrm{II} = \mathrm{II}_o\left(\frac{k}{k_o}\right)^n \qquad (25)$$

where $n = 2(1 - C_\lambda)/C_\lambda$. In Eq. (25), k_o and II_o are reference values taken as the first data point for which the gradient of the mean field has been effectively removed. Parks (1997) concluded that $C_\lambda \cong 2/3$ by using return-to-isotropy data of Choi and Lumley (1984) and of LePenven *et al.* (1985). This phenomenon has also been explained by assuming that the pressure-strain rate correlation in the second-order moment equation for the Reynolds stress is proportional to the anisotropic stress (Launder *et al.*, 1975). A value of $C_\lambda = 2/3$ in the foregoing APS-theory is equivalent to a Rotta constant of three (*cf.* Speziale, 1991).

HOMOGENEOUS SHEAR

For homogeneously sheared turbulence, $\nabla\langle\mathbf{u}\rangle = \dot{\gamma}\,\mathbf{e}_y\mathbf{e}_z$ and $\dot{\gamma}$ = constant (see Figure 1). With $\langle\mathbf{u}'\mathbf{u}'\rangle = 2k\underline{\underline{\mathbf{R}}}$,

$$\underline{\underline{\mathbf{R}}} \equiv R_{xx}\mathbf{e}_x\mathbf{e}_x + R_{yy}\mathbf{e}_y\mathbf{e}_y = R_{zz}\mathbf{e}_z\mathbf{e}_z = R_{yz}\mathbf{e}_y\mathbf{e}_z = R_{zy}\mathbf{e}_z\mathbf{e}_y \quad (26)$$

The preclosure theory defined by Eqs. (7) and (10) gives the following relationships between the components of the Reynolds stress and the components of the prestress (Parks et al., 1998):

$$R_{xx} = \frac{1}{3}\frac{\alpha}{k} + H_{xx} \qquad (27)$$

$$R_{yy} = \frac{1}{3}\frac{\alpha}{k} + H_{yy} \qquad (28)$$

$$R_{yz} = -N_R R_{yy} + H_{yz} \quad (29)$$

The other component equations for the Reynolds stress follow from $\underline{\underline{\mathbf{R}}} = \underline{\underline{\mathbf{R}}}^T$ and $\mathrm{tr}(\underline{\underline{\mathbf{R}}}) = 1$. The first- and second-normal stress differences are given by

$$R_{zz} - R_{yy} = 1 - \frac{\alpha}{k} + H_{zz} - H_{yy} \quad (30)$$

$$R_{yy} - R_{xx} = H_{yy} - H_{xx} \quad (31)$$

The above equations show that the primary role of the isotropic prestress coefficient α is to redistribute the kinetic energy of turbulent fluctuations among the velocity components subject to the normalization requirement that $\mathrm{tr}(\underline{\underline{\mathbf{R}}}) = 1$.

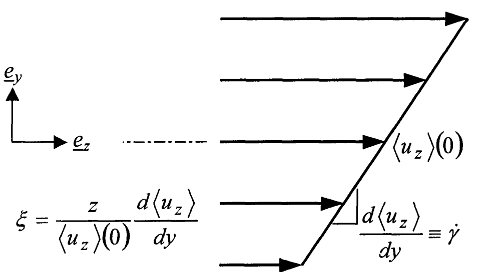

Figure 1. Flow Configuration of Homogeneously Sheared Turbulence.

For homogeneous shear, Eq. (13) reduces to

$$\frac{\alpha}{k} = 1 + 2N_R R_{yz} + N_R^2 R_{yy} \quad (32)$$

where $N_R = \tau_v \dot{\gamma}$. An equation for N_R follows directly from Eqs. (17) and (18) by neglecting the mixing of turbulent kinetic energy and of turbulent dissipation by molecular and turbulent processes. With $\xi \equiv z\dot{\gamma} / < u_z > (0)$,

$$\frac{dN_R}{d\xi} = 2N_R R_{yz}(C_P - 1) + C_R(C_D - 1) \quad (33)$$

The APS-theory presented here assumes that Eq. (20) governs the anisotropic prestress. With $\underline{\underline{\tilde{\mathbf{H}}}} \equiv 2k\underline{\underline{\mathbf{H}}}$, $q \equiv \lambda \frac{1}{k} < \frac{D}{Dt} > k$, and $De_t \equiv \lambda\dot{\gamma}$, the component equations for H_{xx}, H_{yy}, H_{xy}, and H_{yz} are

$$De_t \frac{dH_{xx}}{d\xi} + (1+q)H_{xx} - \tfrac{2}{3}C_M\, De_t\, H_{yz} = 0 \quad (34)$$

$$De_t \frac{dH_{yy}}{d\xi} + (1+q)H_{yy} + (1+\tfrac{1}{3}C_M) De_t\, H_{yz} = 0 \quad (35)$$

$$De_t \frac{dH_{xy}}{d\xi} + (1+q)H_{xy} = 0 \quad (36)$$

$$De_t \frac{dHyz}{d\xi} + (1+q)H_{yz} + \frac{De_t}{2}(H_{zz} - H_{yy}) + C_M \frac{De_t}{2}(H_{zz} + H_{yy}) = \frac{1}{2} C_\beta \frac{k\dot{\gamma}}{\varepsilon} \quad (37)$$

The equation for H_{yx} is the same as Eq. (36). The other component equations of the anisotropic dyadic follow from $\underline{\underline{\mathbf{H}}} = \underline{\underline{\mathbf{H}}}^T$ and $\mathrm{tr}(\underline{\underline{\mathbf{H}}}) = 0$.

ASYMPTOTIC STATES

Homogeneously sheared turbulence has the interesting feature that N ($\equiv k\dot{\gamma} / \varepsilon$) and the components of $\underline{\underline{\mathbf{R}}}$ remain bounded as the flow develops, although $k \to \infty$ and $\varepsilon \to \infty$. The existence of this asymptotic state implies that

$$\left(\frac{d \ln k}{dz}\right)_{z\to\infty} = \left(\frac{d \ln \varepsilon}{dz}\right)_{z\to\infty} \quad (39)$$

If the mixing terms in Eqs. (17) and (18) are neglected, then Eq. (33) predicts the existence of an asymptotic state for N_R ($\to N_R^a < \infty$) provided the ratio of turbulence production to turbulence dissipation remains bounded (Parks, 1997):

$$\left(\frac{- < \mathbf{u}'\mathbf{u}' >: \nabla < \mathbf{u} >}{\varepsilon}\right)_{z\to\infty} = \frac{C_D - 1}{C_P - 1} \quad (40)$$

Thus, with $q \to q^a < \infty$, $De_t \to De_t^a < \infty$, and $N_R \to N_R^a < \infty$, Eqs. (34)-(37) imply that (see Parks, 1997) $H_{xy}^a = H_{yx}^a = 0$, and

$$H_{xx}^a = \frac{2}{3} C_M \frac{De_t^a}{1+q^a} H_{yz}^a \quad (41)$$

$$H_{yy}^a = -(1 + \frac{1}{3}C_M) \frac{De_t^a}{1+q^a} H_{yz}^a \quad (42)$$

$$H_{yz}^a = \frac{C_\beta}{2} \frac{k\dot{\gamma}}{\varepsilon} \frac{(1+q^a)}{(1+q^a)^2 + (1 - \frac{1}{3}C_M{}^2)(De_t^a)^2} \quad (43)$$

If $C_\beta = 0$, then the asymptotic state approaches the IPS result inasmuch as $\underline{\underline{\mathbf{H}}}^a = \underline{\underline{\mathbf{0}}}$. For $C_\beta \neq 0$, Eqs. (31), (41), and (42) imply that

$$R_{yy}^a - R_{xx}^a = -(1 - C_M) \frac{De_t^a}{1+q^a} H_{yz}^a \quad (44)$$

Thus, the second normal stress difference is nonzero provided $C_\beta \neq 0$ and $C_M \neq 1$.

The statistical properties of homogeneous shear measured by Tavoularis and Karnik (1989) imply that $(k\dot{\gamma}/\varepsilon)^a \cong +4.16$, $R_{yz}^a \cong -0.165$, $R_{xx}^a \cong +0.236$, and

$R^a_{yy} \cong +0.196$ (see Parks *et al.*, 1998). The asymptotic second and third invariants of $\underline{\underline{\mathbf{b}}}$ consistent with these results are $II^a \cong 0.138$ and $III^a \cong 0.0174$. For $Re_t >> 1$, the APS-theory contains four phenomenological coefficients: C_M, C_λ, C_β, and C_R. The complementary k-ε model also contains four coefficients: C_D, C_P, C_k, and C_ε. The mixing constants, C_k and C_ε, are important for inhomogeneous flows (see Weispfennig *et al.*, 1999), but do not play a role in homogeneous shear flows

The application of the foregoing theory to isotropic and anisotropic decay experiments implies that $C_D \cong 1.83$ and $C_\lambda \cong 2/3$. Eq. (40) is consistent with the asymptotic result $(-<u'_y u'_z> \dot{\gamma}/\varepsilon)^a \cong +1.37$, provided $C_P \cong 1.60$. The three remaining APS model coefficients were estimated by using Eqs. (27)-(29), Eqs. (41)-(43), and the asymptotic data of Tavoularis and Karnik (1989) with the result that $C_M = -2/3$, $C_\beta \cong +0.174$, and $C_R \cong 0.271$ (Parks, 1997). Table 1 gives a summary of these estimates.

Table 1. Parameter Estimates for the APS Theory.

Parameter	Estimate	Basis
C_D	1.83	Isotropic decay (Comte-Bellot and Corrsin, 1971; Mansour and Wray, 1994; Parks, 1997)
C_P	1.60	Existence condition for *k*-ε equations (Parks, 1997)
C_λ	2/3	Return to isotropy data (*III* > 0; Choi and Lumley, 1983; LePenven, *et al.*, 1985; Parks, 1997)
C_R	0.271	Reproduction of asymptotic state for homogeneous shear (Tavoularis and Karnik, 1989; Parks, 1997)
C_β	0.174	
C_M	-2/3	

TRANSIENT STATES

Figures 2 and 3 illustrate the transient response of N_R and the components of $\underline{\underline{\mathbf{R}}}$, respectively. The transient calculations assume that an initially isotropic, homogeneous turbulence state is subjected to an instantaneous increase in $\dot{\gamma}$. For $\xi < 0$, $N_R = 0$. For $\xi = 0$, $N_R = N^a_R > 0$. The Reynolds stress and the prestress are equal and isotropic for $\xi < 0$. At $\xi = 0$, however, $\underline{\underline{\mathbf{R}}}$ responds instantaneously to the sudden increase in $\dot{\gamma}$ whereas $\underline{\underline{\mathbf{H}}} = \underline{\underline{\mathbf{0}}}$ for $\xi = 0$. Eqs. (7), (10), and (32) are used to calculate the initial condition for the Reynolds stress. With $N^o_R = C_R k \dot{\gamma}^o / \varepsilon$,

$$\underline{\underline{\mathbf{R}}}^o = \frac{\alpha^o}{3k} \underline{\underline{\mathbf{A}}}_o^T \cdot \underline{\underline{\mathbf{A}}}_o \qquad (45)$$

where the operator $\underline{\underline{\mathbf{A}}}_o$ is defined by Eq. (8) with $N_R = N^o_R$. Because $\underline{\underline{\mathbf{H}}}^o = \underline{\underline{\mathbf{0}}}$, it follows from Eq. (36) that $H_{xy} = 0$ for $\xi > 0$.

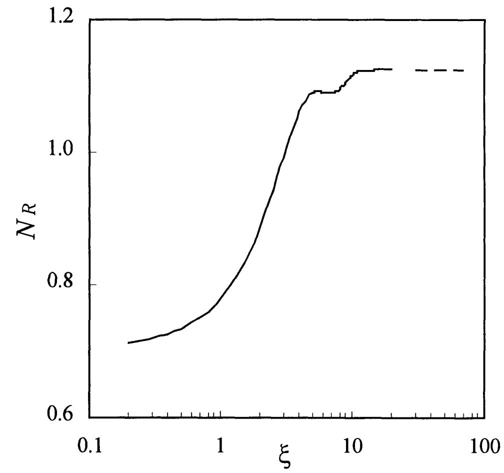

Figure 2. Response of the Relaxation Group to a Sudden Increase in the Mean Strain Rate ($N^o_R = 0.7$; - - -, asymptotic state $N^a_R = 1.13$, Parks *et al.*, 1998).

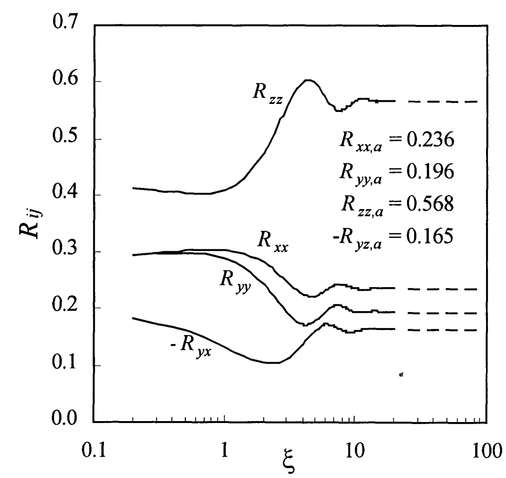

Figure 3. Response of the Reynolds Stress to a Sudden Increase in the Mean Strain Rate ($N^o_R = 0.7$; - - -, asymptotic state, Tavoularis and Karnik, 1989; Parks *et al.*, 1998).

The transient behavior of the nontrivial components of the anisotropic prestress and the preclosure relaxation group N_R were determined numerically by using a fourth-order Runge-Kutta integration algorithm (Carnahan *et al.*, 1969). The initial state characterized by $N^o_R = 0.7$ was selected to closely approximate the initial state of the experimental data slightly downstream of the imposed mean shear field (*i.e.*, $II^o \cong 0.0936$ and $III^o \cong 0.00626$). The calculations show that the asymptotic state obtains for $z \cong 20 < u_z > (0)/\dot{\gamma}$, a theoretical prediction consistent with the previously reported IPS-theory (see Parks *et al.*, 1998) and with experimental observations.

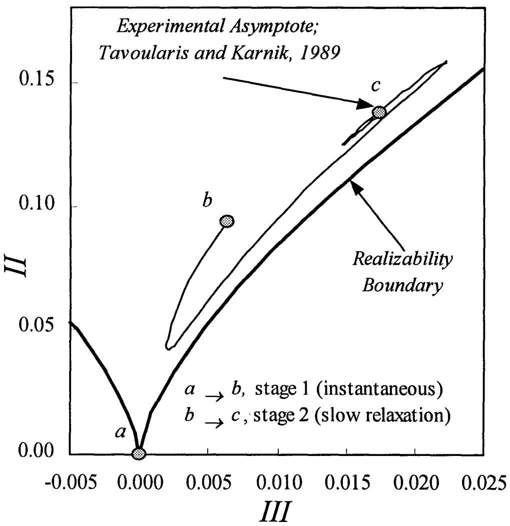

Figure 4. Response of Isotropic Turbulence to a Sudden Increase in the Mean Strain Rate ($N_R^o = 0.7$).

CONCLUSIONS

The isotropic and anisotropic parts of the prestress model represent two distinct responses of the turbulence to a sudden change in the local mean field. The isotropic portion of the prestress causes an instantaneous response to the mean shear rate, immediately reorganizing the Reynolds stress to an anisotropic state consistent with the IPS-theory and the local statistical state of turbulence. Subsequently, the components of the anisotropic prestress relax towards their asymptotic values.

Figure 4 summarizes the transient response in terms of the second and the third invariants of the anisotropic stress. The experimental data of Tavoularis and Karnik (1989) are also shown on Figure 4. On a very short time scale (*i.e.*, $0 < \xi < 1$), the turbulence relaxes back towards an isotropic state in response to the maldistribution of energy caused by the initial response. However, this tendency towards isotropy is reversed by the relatively slow development of the anisotropic prestress and second normal stress difference. The turbulent Deborah number, $\lambda\dot{\gamma}$, controls the relaxation of the Reynolds stress towards an anisotropic asymptotic state.

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