ON ANALYTICAL SOLUTIONS OF TURBULENT PLANE COUETTE FLOW WITH GRADIENT DIFFUSION MODELS

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ABSTRACT

A number of gradient-diffusion models of turbulence predict a constant value for the kinetic energy of turbulence in the central region of plane Couette flow, in clear contrast with experimental and DNS results. It is shown in this paper that the gradient-diffusion approach is responsible for this result through a realizability condition of no counter-gradient diffusive transport.

In the case of the k-ε model, it is shown the proportionality factor, C_μ, used in the definition of the eddy viscosity, is not a constant but a function of the strain rate parameter S^* (Hallbäck et al, 1996) and of the production-dissipation ratio $\lambda = P/\varepsilon$. By assuming λ to depend only on of the strain rate parameter, C_μ may be expressed as a function of S^*, in a way similar to that of Shih et al (1995).

Henry & Reynolds (1984), through an analytical solution, have concluded that two gradient-diffusion models really predict the constant energy value. It is shown here that the aforementioned solution is incorrect and that, on the other hand, a correct analytical solution may be obtained if the right level for the value of C_μ is adopted.

INTRODUCTION

It is now rather well known that a number of gradient-diffusion models of turbulence (Schneider, 1989), including standard k-ε model, anisotropic k-ε models (see e.g. Nisizima & Yoshizawa, 1987), both algebraic and differential Reynolds stress models, with and without wall reflection terms , and even third-order closure models (see Amano & Goel, 1987) predict a constant value for the turbulent energy in the fully turbulent central region of plane Couette flow.

In the present paper, it is shown that the gradient-diffusion approach for the turbulent flux of kinetic energy of the turbulence implies a realizability condition of no counter-gradient diffusive transport, and that the constant value of the turbulent kinetic energy is a consequence of this realizability condition.

In the case of the k-ε model, this realizability condition reduces to a condition for the minimum value of the kinetic energy of the turbulence which is directly related to the value of C_μ, the proportionality factor used in the definition of the eddy viscosity, being normally taken as the constant value 0.09. This value has been adjusted mainly by means of data from pressure driven shear flow experiments like channel flow. In this paper, it is shown that C_μ cannot be a constant and, by means of the experimental data of El Telbany and Reynolds (1982), the correct estimate of C_μ across the channel is obtained (Fig. 2). This estimate is approximately twice the standard value of 0.09, disclosing by this the reason of k-ε model of tending to predict values of ε smaller than P which force the kinetic energy of turbulence to become constant.

Henry & Reynolds (1984) have shown that it is possible to obtain an analytical solution with two different gradient-diffusion models applied to turbulent Couette flow. Unfortunately, the problem was incorrectly solved, and the conclusion that the solution requires the kinetic energy of turbulence to be constant in the fully turbulent central region of the flow is not valid. In the present paper, it is shown that a solution different from the trivial constant-energy solution is possible provided that correct level for the value of C_μ is adopted. The geometrical definitions of the flow for the analysis to follow are given in Figure 1.

NO COUNTER-GRADIENT TRANSPORT CONDITION

The transport equation of turbulent kinetic energy, k, is given by (see e. g. Hallbäck et al, 1996)

$$\frac{Dk}{Dt} = P - \varepsilon - \frac{\partial}{\partial x_m}\left(J_m - \nu \frac{\partial k}{\partial x_m} \right), \qquad (1)$$

where P is the production rate of kinetic energy, ε its dissipation rate, J_m its turbulent transport flux and $\nu \partial k / \partial x_m$ its molecular transport flux. The turbulent transport flux term is usually modelled as having a form similar to that of the molecular transport flux term through a scalar or tensor eddy diffusivity (see e. g. Hallbäck et al, 1996). Owing to the fact that in the case of turbulent plane Couette flow the tensor eddy diffusivity adopts a scalar form, only a scalar eddy diffusivity will be considered in the analysis to follow.

Multiplying the form of equation (1) corresponding to the case of turbulent plane Couette flow by the sum of the modelled and molecular transport fluxes, and integrating between the channel center-line and an arbitrary vertical position within the fully turbulent core of the flow, gives

$$\left(\left(\nu + \frac{\nu_T}{\sigma_k} \right) \frac{dk}{dy} \right)^2 = \int_{k_0}^{k(y)} 2\left(\nu + \frac{\nu_T}{\sigma_k} \right)(\varepsilon - P) dk, \qquad (2)$$

where ν_T is the eddy diffusivity, σ_k the Prandtl-Schmidt coefficient and k_0 the value of k at the channel center-line.

Actually, $(\nu + \nu_T / \sigma_k)$ is fairly constant within the fully turbulent core of the flow, and (dk/dy) is zero at the channel center-line and monotonously increasing with distance from the center-line (see El Telbany & Reynolds, 1982). Thus, the left hand-side of (2) may be integrated within the fully turbulent core of the flow, between two arbitrary vertical positions, the first closer to the channel center-line, and the result should always be greater than zero.

Then, according to the gradient-diffusion model approximation, the rate of dissipation ε has to be grater than or, at least, equal to the rate of production P, at each point within the fully turbulent core of the flow. If the turbulence model tends to predict values of ε less than the corresponding P values, condition (2) forces ε to be equal to P, implying that $(dk/dy) = 0$ and $k = \text{constant}$, since otherwise (dk/dy) or ν_T, or both, should have to become complex. That ε should be greater or at least equal to P within the fully turbulent core of the flow, seems well born out by the DNS results of Andersson et al (1992) and Komminaho et al (1996). Unfortunately, these DNS results have been obtained for rather low Reynolds numbers ($y^+ < \sim 100$ at the channel center-line) and the relatively high Reynolds number experimental results of El Telbany and Reynolds (1982) did not include any estimate of the dissipation rate ε . Even if there are strong indications of the non-existence of counter-gradient diffusion transport for higher Reynolds numbers (Andersson et al, 1992), the issue is still not completely settled. The work of Pattisson et al (1999) with LES applied to plane Couette flow constitute the first stage in an attempt to simulate the flow at higher Reynolds numbers and be able to elucidate this matter.

THE VARIABLE NATURE OF C_μ

The proportionality factor, C_μ, used in the definition of the eddy viscosity of the k-ε model, has normally been taken as a constant, with a value of 0.09 adjusted by means of data from pressure driven shear flow experiments like channel flow. Using the definition of the production rate P, i. e.

$$P = 2\nu_T S_{ij} S_{ij} = C_\mu \varepsilon \left(\frac{k^2}{\varepsilon^2} 2 S_{ij} S_{ij} \right) = C_\mu \varepsilon (S^*)^2, \qquad (3)$$

where S_{ij} is the mean strain rate tensor, $S^* = k\sqrt{2S_{ij}S_{ij}} / \varepsilon$ the strain rate parameter, C_μ may be obtained as

$$C_\mu = \frac{\lambda}{(S^*)^2}, \qquad (4)$$

with $\lambda = P / \varepsilon$ as the production-dissipation ratio. By means of the experimental data of El Telbany and Reynolds (1982), and assuming at each point across the channel the same production-dissipation ratio λ as in Andersson et al (1992), an estimate of C_μ has been obtained and is shown in Figure 2. The corresponding C_μ estimate for the case of plane Poiseuille flow, based on the DNS data of Kim et al (1987) and the experimental data of Laufer (1951), is also shown in the same figure. As may be observed from this figure, C_μ is not constant across the channel, being the estimate for Couette flow almost twice the standard value of 0.09. Even the estimate for Poiseuille flow displays large variations across the channel, being less than half the standard value across a large region of the channel.

The experimental estimates of λ and S^* for Couette and Poisueille flows based on the aforementioned experimental and DNS data are given in Figure 3. The doted curves correspond to S^* (upper curve) and λ (lower curve) for plane Poiseuille flow. The dashed curves correspond to S^* (upper curve) and λ (lower curve) for plane Couette flow. The inspection of the curves for each flow suggests that they are, for values not close to zero, proportional to each other over a large portion of the channel, except at the region close to the walls where their relationship should be determined by the wall conditions. Due to the fact that the models analysed in this paper are high Reynolds number models, the proportionality assumption seems enough accurate. For values close to zero, a quadratic dependence of λ on S^* seems appropriate. Here, the following relationship is suggested

$$\lambda = A_1 S^* \exp\left(A_2 S^* + A_3 (S^*)^2 \right), \qquad (5)$$

where $A_2 = -1/3$, $A_2 = -0.05$ for both flows and $A_1 = 1.83/3$ for Couette flow and $A_1 = 1/3.1$ for Poiseuille flow. The different values of the coefficient A_1 reflect the fact that proportionality factors between λ and S^* are different for the two flows, as may be observed from Figure 3. If no such a difference with coefficient A_1 is made, and the proportionality factor chosen is that for plane Poiseuille flow, C_μ will be underpredicted for plane Couette flow, as is the case with the model of Shih et al (1995). In other words, the proportionality factor depends on the presence of a pressure gradient and λ cannot be modelled only by a dependence on the strain rate parameter S^*.

Finally, C_μ adopts the following form as function of S^*

$$C_\mu = \frac{A_1 \exp\left(A_2 S^* + A_3 (S^*)^2\right)}{S^*}. \qquad (6)$$

ANALYTICAL SOLUTION OF k-ε MODEL

In the case of standard k-ε model, the aforementioned realizability condition reduces to the following condition for the kinetic energy of the turbulence, when ν is neglected in relation to ν_T and k is non-dimensionalised with the friction velocity u_*,

$$k^+ \geq \frac{1}{\sqrt{C_\mu}}. \qquad (7)$$

If the standard value $C_\mu = 0.09$ is considered, condition (7) gives $k^+ \geq 10/3$. Both experiments (El Telbany & Reynolds, 1982, Bech et al, 1995) and DNS results (Andersson et al, 1992, Bech et al, 1995, Komminaho et al, 1996) indicate that k should be smaller than this minimum value in the central region of the flow. This implies that values of ε smaller than the corresponding P values tend to be predicted but cannot be obtained by the model. Therefore, no other alternative than the prediction of a constant kinetic energy level, with the minimum value given by (3), is left for the model. However, this problem may be simply resolved by adopting the correct level of C_μ given in Figure 2.

Two of the aforementioned gradient-diffusion models were analytically analysed by Henry & Reynolds (1984). In general, the analysis is even valid for other gradient-diffusion models using tensor eddy diffusivity since the tensor diffusivity is reduced to a scalar form when applied to plane Couette flow. Unfortunately, the exponents of the Frobenius expansion for dk^+/dy (y non-dimensionalised with the channel half-width h) obtained by Henry & Reynolds (1984), i. e. 7/8 and 5/8, were not the correct values. It is not difficult to prove, for instance by means of the symbolic computation system Maple V (Heal et al, 1996), that the correct exponents are instead 1 and 1/2.

An alternative, easier way of obtaining the analytical solution for this case is to solve for ε^+, non-dimensionalised with the friction velocity u_* and the channel half-width h. The method of Frobenius gives the exponents 1/2 and 0, and the following general expansion for ε^+

$$\varepsilon^+ = \sum_{n=0}^{\infty} a_n \left(k^+ - k_0^+\right)^{n+1/2} + \sum_{n=0}^{\infty} b_n \left(k^+ - k_0^+\right)^n. \qquad (8)$$

ε^+ is related to dk^+/dy by the expression

$$\varepsilon^+ = \frac{C_\mu \left(k^+\right)^2}{\sqrt{g\left(k^+\right)}} \frac{dk^+}{dy}, \qquad (9)$$

where $g\left(k^+\right)$ for the case of the k-ε model is given by

$$g\left(k^+\right) = 2\sigma_k \left(k^+ - k_0^+\right)\left(C_\mu \frac{\left(k^+\right)^2 + k_0^+ k^+ + \left(k_0^+\right)^2}{3} - 1\right). \qquad (10)$$

All the coefficients for $n \geq 1$ in the series expansion (8) may be expressed in terms of a_0 and b_0 (see e. g. Bender & Orszag, 1987). Then, it may be easily shown that if $b_0 \equiv 0$ and $a_0 \neq 0$, all the derivatives of k^+ with respect to y at the channel center-line will be zero, implying that $k^+ = k_0^+$ over the fully turbulent central region of the flow. If, on the other hand, $a_0 \equiv 0$ and $b_0 \neq 0$, all the odd derivatives of k^+ with respect to y at the channel center-line will be zero, implying that $k^+ \neq$ constant over the fully turbulent central region of the flow and symmetric with respect to the channel center-line. From (9) and (10), it is possible to observe that a real solution of the problem will obtained only if condition (7) is satisfied. If the value $C_\mu = 0.16$ is chosen based on the experimental indications shown in Figure 2, the following result, truncated to the 7th power, is obtained if the standard values of the k-ε model constants are used together with $k_0^+ = 2.58$, i. e.

$$\begin{aligned}\varepsilon^+ = b_0 \Big(&1 - 3.61\left(k^+ - k_0^+\right) + 8.45\left(k^+ - k_0^+\right)^2 \\ &- 28.04\left(k^+ - k_0^+\right)^3 + 112.97\left(k^+ - k_0^+\right)^4 \\ &- 503.00\left(k^+ - k_0^+\right)^5 + 2381.67\left(k^+ - k_0^+\right)^6\Big).\end{aligned} \qquad (11)$$

The value of the coefficient b_0 is determined by setting $\varepsilon^+ \approx 10$ for $k^+ = 3.3$, which corresponds to the value of the kinetic energy at $y^+ \approx 100$, i. e. well within the log-region of the wall boundary layer. In this case $b_0 = 0.019$, which also is the value of ε^+ at the channel center-line. The result has been plotted in Figure 4.

CONCLUSIONS

The question of constant kinetic energy at the fully turbulent central region of plane Couette flow being predicted by a number of gradient-diffusion models has been elucidated in the present work by showing that it is originated by a realizability condition inherent to the gradient-diffusion approach.

This realizability condition implies no counter-gradient turbulent transport in the fully turbulent central region of the flow, an issue that is still not completely settled even if strong indications exist that support this assertion (Andersson et al, 1992). The work of Pattisson et al (1999) with LES applied to plane Couette flow constitute the first stage in an attempt to simulate the flow at higher Reynolds numbers and be able to elucidate this matter.

In the case of standard k-ε model, the aforementioned realizability condition has implications in the value of the constant C_μ. By showing the dependence of this "constant" on the strain rate parameter S^* and on the production-dissipation ratio $\lambda = P/\varepsilon$, an experimental estimate of has been obtained indicating that the correct level of C_μ for plane Couette flow is approximately twice that of the standard value. The proportionality between λ and S^* suggested in this paper is not general enough since it depends on the presence of a pressure gradient.

Finally, if this estimated level of C_μ is used together with the correct exponents of the Frobenius expansion, an analytical solution different from the constant-energy value may be obtained in a way similar to that developed by Henry & Reynolds (1984). In the present paper, only the explicit expression for ε^+ has been presented but, there is no doubt that, through relation (9), it is possible to obtain an expression for k^+ which is different from the constant-energy value.

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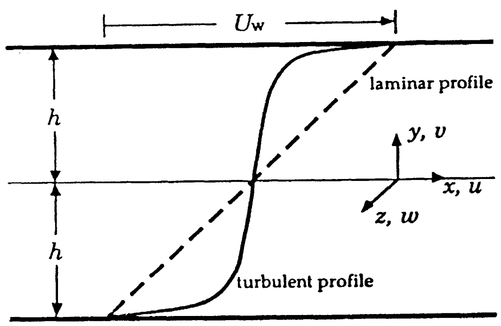

Figure 1. Geometry definition.

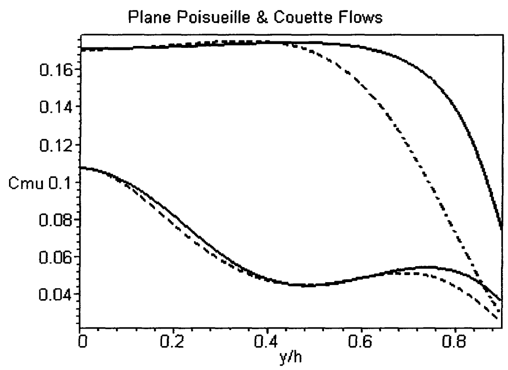

Figure 2. C_μ as a function of S^* across the channel. The upper curves correspond to Couette flow and the lower curves to Poiseuille flow. The solid lines show the approximation according to (6) and the dashed lines the estimate from experiments.

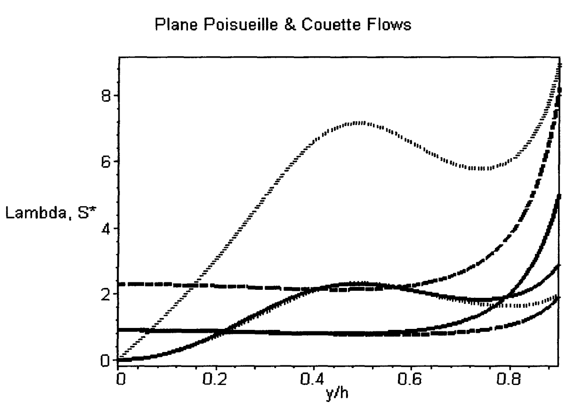

Figure 3. λ and S^* distributions across the channel. Doted curves correspond to S^* (upper curve) and λ (lower curve) for Poiseuille flow. Dashed curves correspond to S^* (upper curve) and λ (lower curve) for Couette flow. Approximation (5) is given by solid lines.

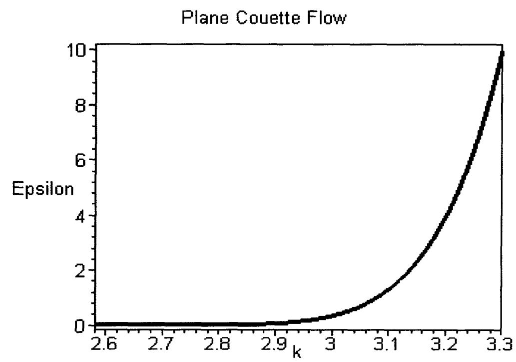

Figure 4. ε as a function of k according to (11).