COMPRESSIBLE TURBULENT FLOW: SYMMETRIES AND SCALING LAWS

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ABSTRACT

Symmetry transformations are one of the most fundamental features of differential equations and illuminate the axiomatic properties of classical mechanics. Therefore we performed a symmetry analysis, also called Lie group analysis (Bluman and Kumei 1989) of the Navier-Stokes equations in the high Reynolds number limit for compressible turbulent flow. The Navier-Stokes equations for compressible flows admit different symmetries than the equations of incompressible flows. These symmetries have to be taken into account for developing models describing compressible turbulence.

An approach to derive turbulent scaling laws based on symmetry analysis is presented in this paper. Starting from the governing equations of compressible turbulent flows we derived their symmetry properties. In the case of isotropic turbulence four symmetry groups were combined. Invariants were computed from the symmetries. These invariants constitute turbulent scaling laws or similarity solutions of the governing equations. Scaling laws were derived for the turbulent kinetic energy in a compressible isotropic flow. These laws describe the decay of turbulence. The analytical solution are compared to results from Direct Numerical Simulation (DNS).

INTRODUCTION

During the investigation of compressible turbulent flows by means of DNS it was realized that a fundamental and comprehensive analysis is needed to account for the compressibility effects of the flow. For isotropic turbulence fluids exhibit a faster decay than for incompressible fluids and additional correlations in the two point correlation equation had to be expressed by analytical methods and compared with DNS. Modelling of the correlations needs a complete understanding of the underlying physical properties of the Navier-Stokes equations for compressible flow.

From the work in Oberlack (1997) on incompressible turbulence it was noticed that symmetry transformations of the Navier-Stokes equations, to be explained below, are one of the key features in order to derive and understand turbulent "scaling laws". It is interesting to note that the Navier-Stokes equation for compressible flows possess different symmetry transformations compared to incompressible flows. Therefore it is expected that different scaling laws can be derived. The main difference to incompressible flows is that symmetry transformations in the compressible domain depend additionally on the form of the equation of state. In the present approach emphasis will be put on modelling implications of the symmetries and the turbulent scaling laws. Any model has to admit exactly the same symmetry properties than the governing unmodeled equations. Since any failure to do so will lead to an unphysical behaviour of the model, for example in an rotating or moving frame of reference.

Symmetry transformations, sometimes simply called symmetries, are transformations which do not alter the structure of the equation under investigation written in the new coordinates. In classical mechanics different types of symmetries are known, e.g. translation in space and time, finite rotation and Galilean invariance to be shown subsequently.

ANALYSIS

Scaling laws can be rigorously derived from symmetry transformation in incompressible turbulence as was shown in Oberlack (1997). Here we report scaling laws which are derived for the compressible turbulence and discuss the results in comparison to DNS.

In order to clarify the analysis below the concept of symmetry transformations, invariants and scaling laws have to be introduced. These are general concepts which can be applied for differential equations to derive exact solutions. The following generic partial differential equation (PDE) is taken as an example to illustrate the concepts.

$$\mathbf{F}(\mathbf{y},\mathbf{z},\mathbf{z}^{(1)},\mathbf{z}^{(2)},...)=0 \tag{1}$$

Here $\mathbf{y}$ and $\mathbf{z}$ represent the independent and the dependent variables respectively and $\mathbf{z}^{(n)}$ refers to all n^{th}-order derivatives of $\mathbf{z}$ with respect to $\mathbf{y}$. A transformation

$$\mathbf{y}=\Phi(\mathbf{y}^*,\mathbf{z}^*) \quad \text{and} \quad \mathbf{z}=\Psi(\mathbf{y}^*,\mathbf{z}^*) \tag{2}$$

is called a symmetry transformation of equation (1) if the following equivalence holds for this transformation

$$\mathbf{F}(\mathbf{y},\mathbf{z},\mathbf{z}^{(1)},\mathbf{z}^{(2)},...)=0 \tag{3}$$
$$\Leftrightarrow \quad \mathbf{F}(\mathbf{y}^*,\mathbf{z}^*,\mathbf{z}^{*(1)},\mathbf{z}^{*(2)},...)=0.$$

Hence the transformation (2) substituted into (1) does not change the functional form of equation (1) if it is written in the new variables $\mathbf{y}^*$ and $\mathbf{z}^*$. Once the symmetry transformation of a partial differential equation are known one can use these to compute self-similar solutions of the PDE.

These concepts are shown in the example of the Euler equations for incompressible fluids

$$\frac{\partial u_i}{\partial x_i}=0 \tag{4}$$

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \tag{5}$$

where u_i is the velocity vector, ρ is the density and p is the pressure.

Among others, equations (4) and (5) admit the following symmetry transformation

$$t^*=\exp(\alpha_2+\alpha_3)t \quad , \tag{6}$$
$$x_i^*=\exp(\alpha_2)x_i \quad , \tag{7}$$
$$u_i^*=\exp(-\alpha_3)u_i \quad , \tag{8}$$
$$p^*=\exp(-2\alpha_3)p \quad ; \tag{9}$$

corresponding to the parameters α_2 and α_3. It is evident that the latter transformations substituted into the equations (4) and (5) preserve their functional form.

The crucial part of understanding similarity solutions in general and the turbulent scaling laws in specific is the concept of invariant functions which will be derived in the subsequent sections.

If the symmetry transformations of equation (1) are known, then a function $f(\mathbf{y},\mathbf{z})$ is called an invariant under transformation (2) if the following condition is met

$$f(\mathbf{y},\mathbf{z})=f(\mathbf{y}^*,\mathbf{z}^*). \tag{10}$$

For the given example of the Euler equations it is obvious that the terms

$$\eta_{x_i}=\frac{x_i}{t^{\frac{a_2}{a_2+a_3}}}, \tag{11}$$

$$\eta_{u_i}=\frac{u_i}{t^{\frac{-a_3}{a_2+a_3}}}, \tag{12}$$

$$\eta_p=pt^{\frac{-2a_3}{a_2+a_3}} \tag{13}$$

are invariants under the transformation (6)-(9) since they do not change their functional form written in the transformed new variables, i.e.

$$\frac{x_i}{t^{\frac{a_2}{a_2+a_3}}}=\frac{x_i^*}{t^{*\frac{a_2}{a_2+a_3}}}, \tag{14}$$

$$\frac{u_i}{t^{-\frac{a_3}{a_2+a_3}}}=\frac{u_i^*}{t^{*-\frac{a_3}{a_2+a_3}}}, \tag{15}$$

$$pt^{\frac{-2a_3}{a_2+a_3}}=p^*t^{*\frac{-2a_3}{a_2+a_3}}. \tag{16}$$

The invariants $\eta_{(...)}$ (11)-(13) of the Euler equations are used as similarity variables that enable the construction of a self-similar solution. If they are introduced into the governing equations they lead to a reduction of the number of independent variables. From the reduced equations it is possible to construct scaling laws, which depend on the arbitrary constants a_2 and a_3.

Each of these coefficients represent one symmetry transformation, e.g. the classical symmetries like translation, rotation or galilean invariance. In the latter example a_2 and a_3 correspond to two scaling symmetries. Further they are needed to adjust for the initial and boundary conditions of the problem. If there is an external scale in the problem, for example caused by a boundary condition, then the variables describing the flow cannot be scaled by the corresponding symmetry transformation since the fixed boundary condition would also be scaled to a new transformed one. Then the arbitrary coefficient of that symmetry transformation can only be zero and the term "breaking of symmetry" is used. In (14)-(16) it can be seen that a rescaling of the velocity is possible if also a proper scaling of time is fulfilled. If there is an external velocity scale in the problem u_i is fixed by $u_i=u_i^*$. Hence from (15) it follows that in this special case the arbitrary constant is fixed to $a_3=0$.

It is possible to investigate at single symmetries seperatly. This was done in the case of the incompressible Euler equations (4) and (5), where only two scaling symmetries were taken into account. We can construct the invariant solutions for more general cases either by combining all the single symmetries of the governing equations or by finding more complex symmetry transformations.

These ideas will be adopted to the equations describing compressible turbulence and invariant solutions of these equations are obtained by using the symmetry transformations.The results will be compared to the scaling law in incompressible turbulence.

Gas Dynamic Equations and its Symmetries

Compressible fluid motion is described by the Navier-Stokes equations. Here we were interested in the turbulent scaling laws of large Reynolds number in the limit $Re \to \infty$ for compressible flows. Hence we begin with symmetry transformations (Ovsiannikov 1962) of the Euler equations as shown below

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0, \tag{17}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0, \tag{18}$$

$$\frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + A(p,\rho)\frac{\mathrm{d}u_i}{\mathrm{d}x_i} = 0. \tag{19}$$

A is specified by

$$A(p,\rho) = -\rho \frac{\partial S}{\partial \rho} \Big/ \frac{\partial S}{\partial p} \tag{20}$$

with the entropy $S = S(p,\rho)$. Following Oberlack (1997 a) and making use of the group theoretical results therein we obtain the following symmetry transformations for compressible turbulence depending on the functional form of A.

If $A(p,\rho)$ is arbitrary the equations (17)-(19) describing compressible flow contain the following symmetry properties:

- translation in time:

$$t^* = t + a_1, \quad x_i^* = x_i, \quad u_i^* = u^i, \quad p^* = p, \quad \rho^* = \rho \tag{21}$$

- translation in space:

$$t^* = t, \quad x_i^* = x_i + a_6, \quad u_i^* = u^i, \quad p^* = p, \quad \rho^* = \rho \tag{22}$$

- scaling of time and space:

$$t^* = t\exp(a_2), \quad x_i^* = x_i\exp(a_2), \quad u_i^* = u^i, \quad p^* = p, \quad \rho^* = \rho \tag{23}$$

and two additional symmetries:

- Galileien
- rotation.

For specific functions of $A(p,\rho)$ these symmetry transformations are extended with additional subsets. For a gas following the ideal gas law $\rho = \frac{p}{RT}$ two additional symmetries are found. Here R is the gas constant and T stands for the temperature. A from (20) is $A = \gamma p$, with γ being the ratio of specific heats.

- scaling of time, velocity and density:

$$t^* = t\exp(a_3), \quad x_i^* = x^i, \quad u_i^* = u_i\exp(-a_3), \quad p^* = p, \quad \rho^* = \rho\exp(2a_3), \tag{24}$$

- scaling of pressure and density:

$$t^* = t, \quad x_i^* = x_i, \quad u_i^* = u_i, \quad p^* = p\exp(a_5) \quad \rho^* = \rho\exp(a_5) \tag{25}$$

In the special case of an mono-atomic gas $\gamma = \frac{5}{3}$ we obtain an additional nonlinear symmetry:

- projection symmetry:

$$t^* = \frac{t}{1-a_4 t}, \quad x_i^* = \frac{x_i}{1-a_4 t}, \quad u_i^* = u_i + a_4(x_i - u_i t), \quad p^* = p + (1-a_4 t)^{(n+2)} \quad \rho^* = \rho + (1-a_4 t)^n \tag{26}$$

Where $n = 3$ stands for the number of dimensions.

Using the theory of approximate symmetries it was observed that the projection symmetry is not broken for small disturbances of γ around its value of $\gamma = \frac{5}{3}$. This fact is assumed to expand the theory to a wider range of application even if non mono-atomic gas is considered.

The above symmetries and their combination can be used to rescale the problem keeping the governing equations unchanged. All the above transformations leave the governing equations invariant. Also a combination of all four constitutes a symmetry of (17)-(19). The order of combination does not matter since the transformation can always be brought into an equivalent form by manipulating the arbitrary constants. Scaling laws that corresponds to these symmetry transformation will be derived and shown in the following section.

Turbulent Scaling Laws

In this work we restricted ourselves to homogeneous isotropic turbulence. Therefore we skip the Galileien symmetry due to the condition of homogeneity of the flow which means that the ensemble averaged velocity U_i is not a function of space.

$$U_i \neq f(x_i) \tag{27}$$

where the averaging relation between the velocity u_i, its fluctuation u_i' and the mean velocity U_i is

$$u_i = U_i + u_i'. \tag{28}$$

Due to the above assumptions of isotropy the rotation symmetry is implied for the flow. Hence

$$\overline{u'^2_1} = \overline{u'^2_2} = \overline{u'^2_3} \tag{29}$$

for the turbulent normal stresses. In the case of isotropic turbulence U_i=0 and hence $u_i = u_i'$.

By combining the symmetry transformations from the latter chapter and keeping in mind that Galilean and rotational symmetries are implied in the velocity field, we obtain the symmetry properties for compressible turbulent flow, which are analysed in this chapter. The translation in time (21), scaling of time and space (23), scaling of pressure and density (25) and the projection symmetries (26) remain in the special case of isotropic turbulence.

The functional form of the projection symmetry (26) shows a spatial dependency of the velocity. If we only consider this symmetry transformation like mentioned above than we obtain the transformation symmetry for the instantaneous velocity

$$u_i^* = u_i(1 - a_4 t) + a_4 x_i. \tag{30}$$

If we ensemble average over the latter equation we obtain

$$U_i^* = U_i(1 - a_4 t) + a_4 x_i. \tag{31}$$

The difference of the equation (30) and (31) yields

$$u'^*_i = u'_i(1 - a_4 t). \tag{32}$$

Hence we have to conclude that the spatial coordinate in the projection symmetry is not of importance for the transformation properties of the fluctuating velocity.

We will combine the two scaling, translation in time and projection symmetry transformations and do obtain two scaling laws for the fluctuating velocity. These laws do depend on the sign of a combination of the arbitrary constants a_i. This is caused by the nonlinear term in the projection symmetry (26). We then define

$$\Delta = 4a_4 a_1 - (a_2 + a_3)^2, \tag{33}$$

where the a_i's represent the listed symmetries above.

Using the combination of the symmetries (21), (23), (24) and (26) we obtain the following symmetry transformations for $\Delta > 0$

$$t^* = \frac{\sqrt{\Delta}\tan(\frac{1}{2}\sqrt{\Delta}) + 2a_4 t + a_3 + a_2}{\sqrt{\Delta} - \tan(\frac{1}{2}\sqrt{\Delta})(2a_4 t + a_3 + a_2)} \frac{\sqrt{\Delta}}{2a_4} - \frac{a_3 + a_2}{2a_4} \tag{34}$$

$$u^* = u e^{\frac{a_2 - a_3}{2}} \frac{\sqrt{\Delta} - (2a_4 t + a_3 + a_2)\tan(\frac{\sqrt{\Delta}}{2})}{\sqrt{\Delta}\sqrt{1 + \tan(\frac{1}{2}\sqrt{\Delta})^2}}$$

and for $\Delta < 0$ the combination of symmetry transformations gives

$$t^* = \frac{\sqrt{-\Delta}\tanh(-\frac{1}{2}\sqrt{-\Delta}) + 2a_4 t_+ a_3 + a_2}{\sqrt{-\Delta} + \tanh(-\frac{1}{2}\sqrt{-\Delta})(2a_4 t + a_3 + a_2)} \frac{\sqrt{-\Delta}}{2a_4} \tag{35}$$

$$- \frac{a_3 + a_2}{2a_4} \tag{36}$$

$$u^* = u e^{\frac{a_2 - a_3}{2}} \frac{\sqrt{-\Delta} - (2a_4 t + a_3 + a_2)\tanh(-\frac{\sqrt{-\Delta}}{2})}{\sqrt{-\Delta}\sqrt{1 + \tanh(-\frac{1}{2}\sqrt{-\Delta})^2}}$$

There are two invariant functions for the velocity that remain unchanged under the symmetry transformation (34) and (36) in the new transformed coordinates t^* and u^* as mentioned in the analysis section. They are distinguished by the sign of Δ. For a positive sign of Δ we obtain

$$u' = C_1 \frac{e^{\frac{a_2 - a_3}{\sqrt{\Delta}} \arctan \frac{2a_4 t + a_2 + a_3}{\sqrt{\Delta}}}}{\sqrt{a_4 t^2 + (a_2 + a_3)t + a_1}}, \tag{37}$$

while in the case of $\Delta < 0$ the decay of u follows the law

$$u' = C_2 \left(\frac{2a_4 t + a_2 - \sqrt{-\Delta}}{2a_4 t + a_2 + \sqrt{-\Delta}}\right)^{\frac{a_2 - a_3}{2 * \sqrt{-\Delta}}} * \; * \frac{1}{\sqrt{a_4 t^2 + (a_2 + a_3)t + a_1}}, \tag{38}$$

or equivalent

$$u' = C_3 \frac{e^{\frac{a_2 - a_3}{\sqrt{-\Delta}} \tanh^{-1} \frac{2a_4 t + a_2 + a_3}{\sqrt{-\Delta}}}}{\sqrt{a_4 t^2 + (a_2 + a_3)t + a_1}}, \tag{39}$$

In both cases the result is a decay law for the turbulent kinetic energy proportional to

$$k \sim t^{-2} \tag{40}$$

in the limit of large Reynolds number and $t \to \infty$.

If a_4 is set to zero the transformation symmetries reduce to the symmetries of the equations for incompressible flows. Hence we obtain the well know result for the decay of the velocity fluctuations in incompressible turbulence

$$u = C_1[a_1 + (a_2 + a_3)t]^{-\frac{a_2}{a_2 + a_3}}, \tag{41}$$

adopted in almost all turbulence models. Our interest is now focused on the case $a_4 \neq 0$.

In Figure (1) DNS results for the decay of the turbulent kinetic energy of an incompressible and a compressible flow at a turbulent Mach number $M_t = \frac{\sqrt{u_i' u_i'}}{\bar{c}} = 0.4$ are shown, where $\bar{c}$ is the mean speed of sound. The compressible flow simulation exhibits a faster decay than the incompressible one, which may be induced by the same effects as the higher decay exponent in equation (37) and (38) versus the "incompressible" decay law (41). This comparison is only in a qualitative way since our analysis is carried out in the high Reynolds number range and the DNS was done for a finite "low" Reynolds number.

The present approach can be expanded to take into account a variable Mach number. All group parameters in the corresponding scaling laws to (37) and (38) then do depend on the Mach number. Hence compressibility effects of turbulence naturally come in due to the rigorous symmetry analysis. Work will be continued on asymptotic methods which give information about scaling laws in special parameter regimes of Mach number, and about the stability of these laws with regard to disturbances in these dimensionless parameters.

The results outlined above have serious implication for turbulence models. Because of the outstanding importance of symmetries for classical mechanics, proposed one- and two-point closure models of turbulence need to satisfy the symmetry properties of the underlying unclosed equations to maintain the ability of capturing the proper flow physics. A wide variety of models for incompressible turbulence do not account for all symmetries of the "incompressible" Navier-Stokes equations, as has been shown by Oberlack (1999). However a few models respect the "incompressible" symmetries. Nevertheless it appears that simple extensions of "incompressible models" may not allow for the additional symmetries of compressible flows.

Therefore existing "incompressible" models like the $k-\varepsilon$ or the Smagorinsky model in Large Eddy Simulation (LES) of turbulence may not be easily extendible for compressible flows. New compressible models of Sarkar (1991) and Zeman (1991) take into account additional dissipation and dilatation due to compressibility effects. See Blaisdell (1993) for a discussion on these models. Their calculated decay rate, from asymptotic analysis and DNS for compressible flows is also faster than for the incompressible turbulence. These effects have previously been modelled using heuristic arguments. With the presented symmetry analysis a rigorous derivation from first principles is obtained.

SUMMARY AND CONCLUSIONS

The general purpose of the present work is to establish base for a physical reliable modelling of compressible turbulent flows. A symmetry analysis is proceeded to understand the different behaviour of the compressible flow in comparison to incompressible. The decay rate of homogeneous compressible turbulence is faster than for incompressible turbulence. Also there are new symmetries in the equations for compressible flows from which different scaling laws can be derived.

We derived the invariants which are used to obtain similarity solutions for compressible flows. These invariants reduce the order of the problem and lead to similarity solutions. The obtained analytical results from symmetry transformation theory are compared to DNS and show the same tendency.

An understanding of the effect of Mach number on the symmetry properties is needed. Therefore a combination of asymptotic methods and symmetry transformation concepts is presently conceived. Further there are concepts of approximate symmetries which will be applied to the compressible Navier-Stokes equation as well.

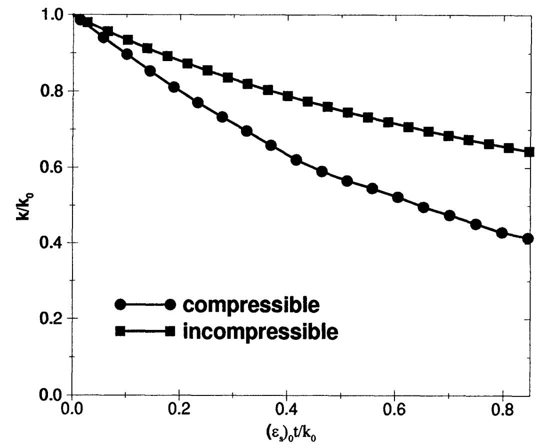

FIGURE 1. Decay of isotropic turbulence for an incompressible vs. a compressible fluid at $M_t = 0.4$

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