# THE PDF PROFILE IN THE LOG-LAW REGION AND THE CONTRIBUTION FROM COHERENT STRUCTURES

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## ABSTRUCT

In a zero-pressure gradient turbulent boundary layer, we made a multi-point measurement by 24 I-probes in an area from 7.0 to 322.0 in wall unit. The log-law region is defined first by the invariant assumption of pdfs which is a new idea to fix the logarithmic velocity extent, and then the relation to the coherent structures are considered by POD technique and stochastic estimation. Coherent structures play an important role to make up the log-law profile and they are almost the same order of log-law region.

## INTRODUCTION

In a zero-pressure gradient turbulent boundary layer, we think about a relation between mean velocity profile and coherent structures by analyzing the experimental data measured in low Reynolds number flow. One of the recent interesting topics is the mean velocity profile in a wall bounded shear flow.(1) So far, the logarithmic profile is considered as a firmly established result and has been believed to be a universal scaling law. However, the recent research cast doubts about its existence, in which the universal scaling is regarded as a power-law distribution.(2) In a low Reynolds number flow, the difference between these two scaling laws is very little, and no clear discussion may be impossible to predict which scaling law is preferable. Therefore, we will not comment this issue but think about the reason why the universal scaling law is realized. If the logarithmic region is a good representation of the experimental data, what the log-law region is indicating? Is there any physical phenomena to predict the scaling law? These are our motivations to start this study.

We assume that the coherent structure is a key factor to think about the extent of logarithmic velocity profile. Proper orthogonal decomposition is used to detect it, and the relation to the log-law region is considered. We have presented the idea to define the log-law region by using the probability density function (pdf) of the normalized streamwise component. Logarithmic region is defined as the extent where the pdf profile remains unique.(3) We call this the invariant assumption of pdfs, which has been confirmed to give a reasonable logarithmic region in moderate Reynolds number flows. Then the question is whether the invariant assumption is broken off by the coherent structure or not.

## EXPERIMENTAL CONDITION

In a wind tunnel with a test section $0.32 \times 1.06m$ in area and $2.6m$ in length, a typical two-dimensional turbulent boundary layer is generated. The data are measured at $1900mm$ downstream from the leading edge with using multi I-type probes. The rake is consist of 24 probes in which each probe is made of tungsten wire with $5\mu m$ in diameter and $1.0mm$ in length, and it is separated by $1.0mm$ with each other. They are located from 7.0 to 322.0 in wall unit. A schematic diagram is shown in Fig.1.

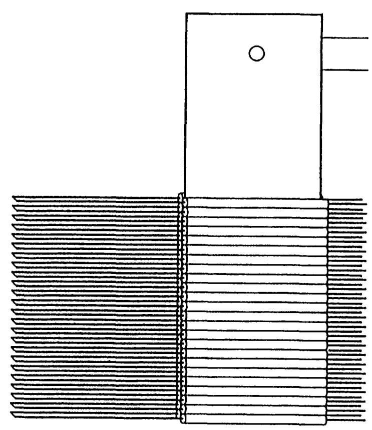

Fig. 1 Schematic view of the rake.

Free stream velocity is $U_0 = 5.0m/s$, nominal boundary layer thickness is $\delta = 40.0mm$, and the Reynolds number based on the momentum thickness is $R_\theta = 1320$. The probes are operated by a handmade constant-temperature anemometer (see Fig. 1(b)), and the analog signal is sampled by 12-bit A/D converter with 5kHz sampling frequency which is linearized by a computer.

The measured mean velocity and the r.m.s. velocity distributions are indicated in Fig.2. The dashed line is a result measured by a single I-type probe in the same experimental condition. They agree well each other.

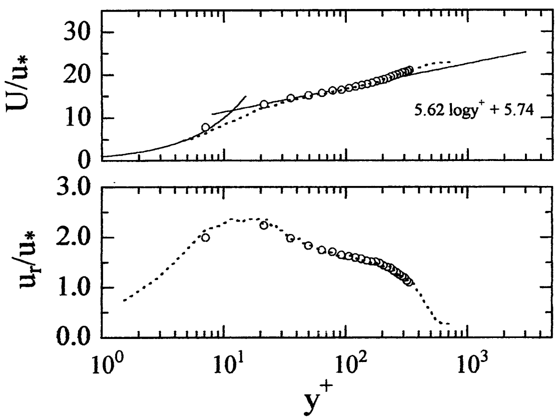


Fig. 2 Mean velocity and the turbulent intensity distribution in this measurement. The dashed line is the result measured by a single probe at the same condition.

## THE PROCEDURE TO EXTRACT THE LOG-LAW REGION

The extent of the logarithmic region is classically considered as an equilibrium region where the total shear stress is constant.[4] But the "equilibrium" is interpreted here as the condition that the pdf profile of the normalized streamwise velocity component does not change. That is, the pdf profile remains unique in the log-law region. We call this idea the invariant assumption of pdf profile.

When the instantaneous velocity in streamwise component is decomposed into mean and fluctuation as $\tilde{u} = U + u'$, we think about the pdf of normalized velocity; $u = u'/u_r$, where $u_r$ is r.m.s. value of $u'$. If the invariant region of the pdf profile exist, we regard this as the log-law region.

In experimental data analysis, the invariant assumption of pdfs is a little relaxed. We extract the region where the pdf has a "similar" profile but not the "same " one. The Kullback Leibler divergence (KLD)[5] is used to distinguish the pdf's profile, which is defined as,

$$D(P\|Q) \equiv \sum_{\{\mathbf{s}\}} P(s_i)\log_e\left(P(s_i)/Q(s_i)\right) , \qquad (1)$$

where $P(\mathbf{s})$ and $Q(\mathbf{s})$, $\{\mathbf{s}\} = \{s_1, s_2, \cdots\}$, are discrete probability distributions. KLD has a non-negative value for any $P(\mathbf{s})$ and $Q(\mathbf{s})$, and it is zero only when $P(\mathbf{s})$ is the same with $Q(\mathbf{s})$. As KLD has a smaller value, then $P(\mathbf{s})$ and $Q(\mathbf{s})$ are more similar. That is, it is a indicator to evaluate how much $P(\mathbf{s})$ resembles $Q(\mathbf{s})$.

A typical example is shown in Fig. 3, in which KLD is computed by the probability distributions obtained at $y$ and $y'$. In the inner region KLD is small and this means that the pdf profiles resemble each other. We introduce the threshold value $\Gamma_{th}$ and find out the region $[y_1^+, y_2^+]$ subject to the condition,

$$D(P_y\|P_{y'}) < \Gamma_{th} \quad , \quad \forall y, \forall y' \in [y_1^+, y_2^+] \ , \qquad (2)$$

where $P_y$ and $P_{y'}$ is the probability distribution measured at $y$ and $y'$, respectively. The threshold is indicating the criterion how much $P_y$ and $P_{y'}$ resembles each other. When $\Gamma_{th} = 0$, probability profiles are exactly the same with each other. However, for a given $\Gamma_{th}$, the extent $[y_1^+, y_2^+]$ is not fixed uniquely.

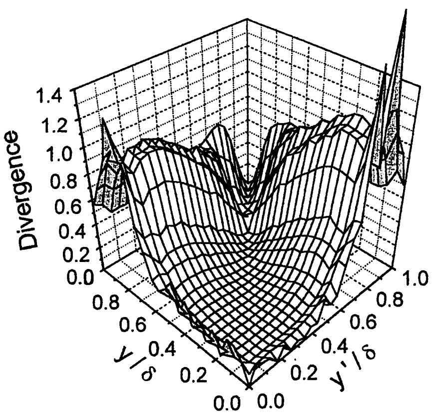


Fig. 3 A typical example of KLD computed by the streamwise velocity component.

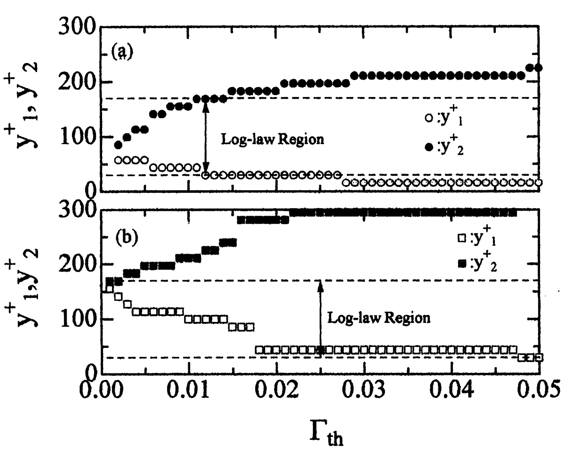


Fig. 4 (a) The extent $[y_1^+, y_2^+]$ defined by Eq. (2) for the original $u$ fluctuation. (b) The extent $[y_1^+, y_2^+]$ defined by Eq. (2) for the reconstructed velocity signal $u^*$ in Eq. (5).

Then we condition that the distance $L = y_2^+ - y_1^+$ has a maximum value. In Fig.4(a) the extent $[y_1^+, y_2^+]$ is plotted against the threshold. When $\Gamma_{th}$ increases, $y_2^+$ also increase, but $y_1^+$ approaches the constant value. Once $y_1^+$ reaches the constant, the log-law region is defined (see Fig.4(a)).[3] The pdf profiles are very different at close to the wall, therefore, $y_1^+$ does not change for large $\Gamma_{th}$. The minimum $y_1^+$ is about 30 in wall unit independent of Reynolds number. The previously obtained results are shown in Fig.5 in which the log-law region is indicated as solid symbols for several Reynolds numbers. The lower end $y_1^+$ is almost constant but the upper end is a function of Reynolds number. The invariant assumption can predict well the reasonable log-law region.

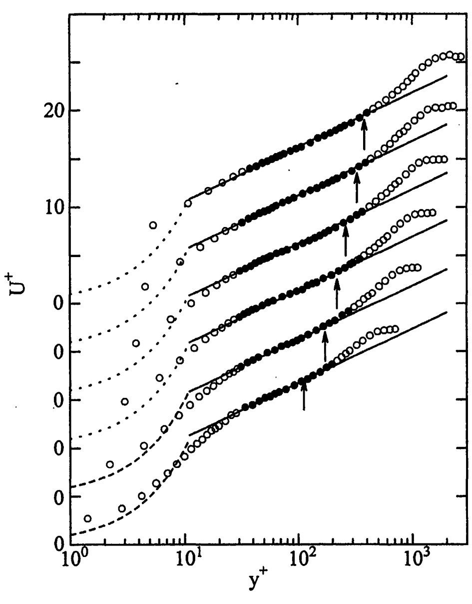


Fig. 5 The log-law region is indicated by a solid symbol which is derived by the invariant assumption of pdfs. The arrows point out the position $y/\delta = 0.2$.

## PROPER ORTHOGONAL DECOMPOSITION

Proper orthogonal decomposition (POD) is an essential tool to identify the coherent structure.[6] This provides a mathematically rigorous procedure for extracting the most energetic modes from a random field. POD has been applied by many researchers to various flow field and the system of nonlinear ordinary differential equations governing the amplitudes of spatial structures has been reported. So the detailed explanations are omitted here. The streamwise velocity fluctuation is decomposed as

$$u'(y,t) = \sum_{i=1}^{N} a_i(t)\phi_i(y)\,, \qquad (3)$$

where $N$ is a total measuring position, $N = 24$. The total energy in the region $\mathbf{I} : 7.0 < y^+ < 322.0$ is given as a summation of the eigenvalues

$$E_T \equiv \int_{\mathbf{I}} u_r(y)^2 dy = \sum_{i=1}^{N} \lambda_i\,, \qquad (4)$$

where $u_r = \sqrt{\langle u'^2 \rangle}$. The most important property is that the convergence of the representation is optimally fast. Each eigenvalue represents the ensemble-averaged energy content in each mode, and the first eigenmode contains the large amount of the total energy. In this analysis, the first eigenmode contains about 40% energy, and the second one has 20%, that is, $(\lambda_1 + \lambda_2)/E_T \simeq 0.6$. The eigenvalue decays in the exponential form like $\lambda_x \propto \exp(-\alpha x)$, and $\alpha = 0.29$ in this system. A typical example of instantaneous velocity field is shown in Fig. 6(a) in which a few large scale structures are contained. The contour indicates the normalized turbulent fluctuation, $u'/U_0$. In Fig. 6(b) this instantaneous velocity field is reconstructed by the first and second POD modes. The coherent large motions are caught well by these two modes.

## RESULTS AND DISCUSSION

We mentioned the procedure to extract the log-law region from the invariant assumption of pdfs. And then the coherent structures are well extracted by a few modes of POD technique. Therefore, in this section we consider the relation between the log-law region and the coherent structures.

The velocity field is reconstructed by POD series without the first and second eigenmodes. That is, from Eq. (3) we have

$$u^*(y,t) = \sum_{i=3}^{N} a_i(t)\phi_i(y)\,, \qquad (5)$$

which does not contain the fluctuation caused by the coherent structures. The invariant assumption of pdf is checked for $u^*$, and the result is indicated in Fig. 4(b). From the beginning of $\Gamma_{th}$, $y_2^+$ is larger than the upper end of the log-law, and when $y_1^+$ reaches the constant value, $y_2^+$ is located far from the wall. This means that the invariant assumption is broken. There is no reasonable region where the pdf's profiles resemble each other for small $\Gamma_{th}$. This is because the coherent structure plays a significant role in the log-law region to produce the turbulence intensity. To confirm this, the stochastic estimation is applied. Using the POD ingenfunction, the velocity at $y'$ is predicted by the velocity at $y$ in an average sense that

$$\langle u'(y')|u'(y)\rangle = \frac{\sum_{i=1}^{N} \lambda_i \phi_i^*(y)\phi_i(y')u'(y)}{\sum_{j=1}^{N} \lambda_j |\phi_j(y)|^2}\,. \qquad (6)$$

The velocity at $y$ is a reference velocity, and $u'(y')$ is predicted by the Eq. (6). In Fig. 5 we indicate the large scale motion which is predicted by the reference velocity at $y^+ \simeq 30$ and $y^+ \simeq 90$, the former is almost the beginning of the log-law region and the latter is the middle point of it. Coherent structures are almost the same scale with the log-law region.

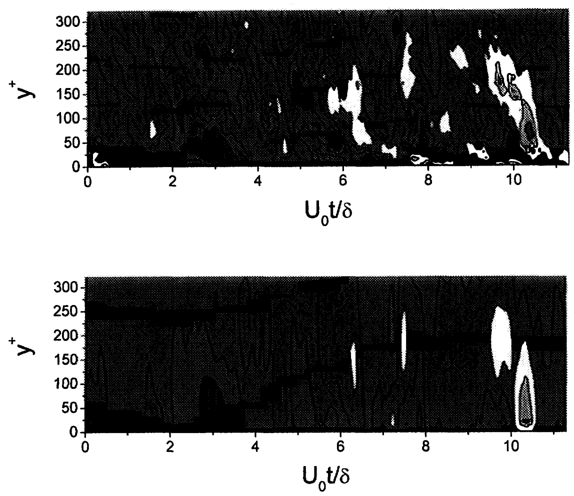


Fig. 6 (a) Instantaneous velocity contour normalized by the outer variable; $u'/U_0$. (b) Reconstructed velocity field by the first and the second POD eigenmodes.

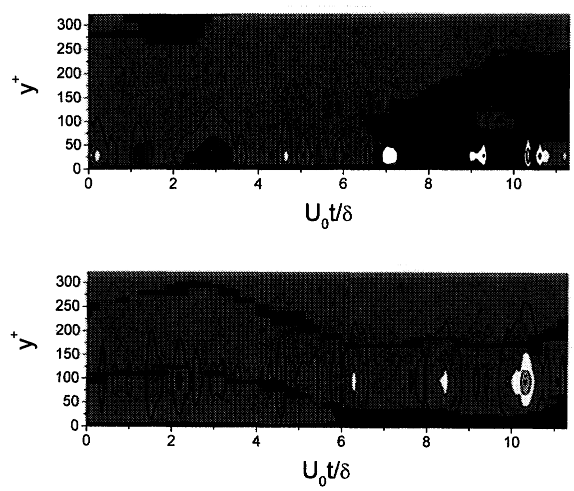


Fig. 7 (a) Large scale motion predicted by the stochastic estimation from the reference point at $y^+ \simeq 30$. (b) Large scale motion predicted by the stochastic estimation from the reference point at $y^+ \simeq 90$.

## CONCLUSIONS

In zero-pressure gradient boundary layers, we present the definition of the log-law region as the extent where the pdf profile of normalized $u$-component velocity fluctuation remains unique. Coherent structures are identified by a few modes of POD technique, and the invariant assumption is checked for the velocity field in which the coherent structures are removed. The invariant assumption is broken off, therefore coherent structures play a significant role in log-law region. Stochastic estimation enables us to know the mean scale of the coherent structures in space. They are almost comparable to the log-law region when the reference point is set at $y^+ \simeq 90$. Even when it is located at $y^+ \simeq 30$, the effect extends up to $y^+ \simeq 120$.

## REFERENCES


**(1)** B. Cipra, "A New Theory of Turbulence Causes a Stir Among Experts", Science, **272**, 951 (1996).

**(2)** G. I. Barenblatt and A. J. Chorin, "Scaling Laws and Vanishing-Viscosity Limits for Wall-Bounded Shear Flows and for Local Structure in Developed Turbulence", Communications on Pure and Applied Mathematics, **50**, 381 (1997).

**(3)** Y. Tsuji and I. Nakamura, "Probability Density Function in the Log-law Region of Low-Reynolds Number Turbulent Boundary Layer", Phys. of Fluids, **11**, 647 (1999).

**(4)** J. C. Rotta, "Turbulent Boundary Layer in Incompressible Flow", Progress in Aeronautical Sciences, **2**, 1 (1962).

**(5)** S. Kullback, "Information Theory and Statistics", John Wiley & Sons, Inc., London, (1959).

**(6)** P. Holmes, J. L. Lumley and Gal Berkooz, "Turbulence, Coherent Structures, Dynamical Systems and Symmetry", Cambridge Univ. Press., (1996).